REMARKS ON GEOMETRY AND THE QUANTUM POTENTIAL

ROBERT CARROLL
UNIVERSITY OF ILLINOIS, URBANA, IL 61801

Abstract. We gather material from many sources about the quantum potential and its geometric nature. The presentation is primarily expository but some new observations relating Q, V, and ψ are indicated.

Contents

1. INTRODUCTION

In [21] we surveyed many feature of the so called quantum potential (QP) (cf. also [22, 23, 24, 25, 26, 27, 28, 29]). Some matters were treated more thoroughly than others and we want to discuss here certain geometrical aspects in more detail, some connections to nonlinear Schrödinger type equations, and various phase space approaches. The latter two topics were not developed in [21] and we will try to make amends here. Some relations to electromagnetic (EM) theory will also be discussed. To set the state we

Date: February, 2007.
email: rcarroll@math.uiuc.edu.

recall the Schrödinger equation (SE) in 1-D of the form (**1A**) $-(\hbar^2/2m)\psi''+V\psi = i\hbar\psi_t$ so that for $\psi = Rexp(iS/\hbar)$ one has

$$(1.1)\quad S_t + \frac{1}{2m}S_x^2 + V + Q = 0;\ Q = -\frac{\hbar^2 R''}{2mR};\ \partial_t(R^2) + \frac{1}{m}(R^2 S_x)_x = 0$$

Here Q is the quantum potential (QP) and one can argue that Bohmian mechanics is simply classical symplectic mechanics using the Hamiltonian (**1B**) $H_q = H_c + Q = (1/2m)S_x^2 + V + Q$ from the Hamiltonian-Jacobi (HJ) equation (1.1) (cf. here [12, 14, 53, 70]). One can write $P = R^2 = |\psi|^2$ (a probability density) with $\rho = mP$ a mass density and obtain a hydrodynamical version of (1.1). Note in particular (**1C**) $Q = -(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho})$ and using $p = S_x = m\dot{q} = mv$ one obtains

$$(1.2)\qquad mv_t + mvv_x + \partial V + \partial Q = 0;\ \rho_t + (\rho\dot{q})_x = 0$$

leading to

$$(1.3)\qquad \partial_t(\rho v) + \partial(\rho v^2) + \frac{\rho}{m}\partial V + \frac{\rho}{m}\partial Q = 0$$

which has the flavor of an Euler equation (cf. [21, 26, 41]). There is however a missing pressure term from the hydrodynamical theory (cf. [21, 76, **?**]) and looking at (1.2) one could imagine a pressure term supplied in the form (**1D**) $\partial Q = (1/R^2)\partial\mathfrak{P}$ (where $\mathfrak{P}$ denotes pressure). This suggests a hydrodynamical interpretation for Q, namely, going to 3-D for example, (**1E**) $\nabla\mathfrak{P} = R^2\nabla Q$ (cf. [26]). This will all be discussed in detail below and we make first a few background remarks about the QP.

REMARK 1.1. In [22] we considered given a function $Q \in L^\infty(\Omega)$ (for Ω a bounded domain) and looked for $R \in H_0^1(\Omega)$ satisfying $Q = -(\hbar^2/2m)(\Delta R/R) \equiv \Delta R + (2m/\hbar^2)QR = 0$. We showed that if $Q < 0$ ($\beta = (2m/\hbar^2)$) then there is a unique solution and if 0 is not in the countable spectrum of $\Delta R + \beta QR$ then $\Delta R + \beta QR = 0$ has a unique solution for any $Q \in L^\infty$. The corresponding HJ equation (**1F**) $\partial_t S + (1/2m)(\nabla S)^2 + Q + V = 0$ and the continuity equation (**1G**) $\partial_t R^2 + (1/m)\nabla(R^2\nabla S) = 0$ must then be solved to obtain some sort of generalized quantum theory. Here a priori V must be assumed unknown and there are then two equations for two unknowns S and V, namely (in 1-D for simplicity)

$$(1.4)\qquad S_t + \frac{1}{2m}S_x^2 + Q + V = 0;\ \partial_t R^2 + \frac{1}{m}(R^2 S_x)_x = 0$$

the solution of which would yield a SE based on Q (see here Remark 5.1 for more detail in this regard). ■

EXAMPLE 1.1. Now $(1/2m)p^2 + V = E$ (classical Hamiltonian - recall $p \sim S_x$) so we could perhaps treat E as an unknown here and try to solve

$$(1.5)\qquad S_t + Q + E = 0;\ \partial_t R^2 + \frac{1}{m}(R^2 S_x)_x = 0$$

Consider first $R^2S_x = -\int^x m\partial_t R^2 dx + f(t)$ from which

$$2RR_tS_x + R^2S_{xt} = -\int^x m\partial_t^2 R^2 dx + f' \Rightarrow (Q_x + E_x)R^2 = \tag{1.6}$$

$$= -R^2S_{xt} = \frac{2R_t}{R}\left(-\int^x m\partial_t R^2 dx + f\right) + \int^x m\partial_t^2 R^2 dx - f'$$

Hence

$$R^2S_x = -\int^x m\partial_t R^2 dx + f(t);\; R^2E_x = -Q_xR^2 + \tag{1.7}$$

$$+\int^x m\partial_t^2 R^2 dx - f' + \frac{2R_t}{R}\left(-\int^x m\partial_t R^2 dx + f\right)$$

giving S_x and E_x modulo an arbitrary differentiable function $f(t)$. Note also

$$R^2E_x = R^2V_x + \frac{R^2}{2m}(S_x^2)_x = R^2V_x + \frac{R^2}{m}S_xS_{xx} \tag{1.8}$$

and S_{xx} can be determined via $(1/m)(R^2S_x)_x = -\partial_t R^2$. Hence R^2V_x can be determined from R^2E_x. Note here

$$\frac{R^2}{m}S_xS_{xx} = -\partial_t R^2 - \frac{2}{m}RR_xf + 2RR_x\int^x \partial_t R^2 dx \tag{1.9}$$

(see Remark 5.1 for more details). ■

We mention also two examples from [12, 22]

EXAMPLE 1.2. For a free particle in 1-D there are possibilities such as $\psi_1 = Aexp[i[px - (p^2t/2m))/\hbar]$ and $\psi_2 = Aexp[-i(px + (p^2t/2m))/\hbar]$ in which case $Q = 0$ for both functions but for $\psi = (1/\sqrt{2})(\psi_1 + \psi_2)$ there results $Q = p^2/2m$ ($p \sim \hbar k$ here). Hence $Q = 0$ depends on the wave function and cannot be said to represent a classical limit. Further we note that $S = \hbar kx - (\hbar^2k^2/2m)t$ in ψ_1 with $S_t = -\hbar^2k^2/2m \sim -E$, $S_x = \hbar k$, and $R = 1 \notin H_0^1$. For $\psi = (1/\sqrt{2})(\psi_1 + \psi_2)$ on the other hand

$$R = \sqrt{2}ACos(kx) \notin H_0^1;\; \frac{R''}{R} = -k^2;\; Q = \frac{k^2\hbar^2}{2m};\; S = -\frac{k^2\hbar^2t}{2m}; \tag{1.10}$$

$$S_t = -\frac{k^2\hbar^2}{2m} \sim -E;\; S_x = 0$$

Thus the same SE can arise from different Q (which is generally obvious of course) and S varies with Q. ■

EXAMPLE 1.3. For $V = m\omega^2x^2/2$ and a stationary SE one has solutions of the form $\psi_n(x) = c_nH_n(\xi x)exp(-\xi^2x^2/2)$ where $\xi = (m\omega\hbar)^{1/2}$, $c_n = (\xi/\sqrt{\pi}2^nn!)$, and H_n is a Hermite function. One computes that $Q = \hbar\omega[n + (1/2)] - (1/2)m\omega^2x^2$ and hence $\hbar \to 0$ does not imply $Q \to 0$ and moreover $Q = 0$ corresponds to $x = \pm\sqrt{(2\hbar/m\omega)[n + (1/2)]}$ so not all systems in

quantum mechanics have a classical limit. This example corresponds to $\Omega = \mathbf{R}$ and $\psi_n \in H_0^1$ is satisfied. ■

2. REMARKS ON WEYL GEOMETRY

Now we recall how in various situations the QP is proportional to a Weyl-Ricci curvature R_w for example (cf. [21, 24, 33, 102]) and this can be interpreted in terms of a statistical geometry for example (cf. also [3, 4]) In general (see e.g. [12, 21, 22]) one knows that each wave function $\psi = Rexp(iS/\hbar)$ for a given SE produces a different QP as in (1.1) (which in higher dimensions has the form $-(\hbar^2/2m)(\Delta R/R)$). Thus for $Q \sim R_w$ to make sense we have to think of a given R or $R^2 = P$ (or $\rho \sim mP$) as generating a (Weyl) geometry as in [102]. (cf. also [21, 24, 33]). This is in accord with having a Weyl vector (**2A**) $\phi_i \sim -\partial_i log(\hat{\rho})$ (where $\hat{\rho} = \rho/\sqrt{g}$ in [102] for a Riemannian metric g). Thus following [102] one assumes that the motion of the particle is given by some random process $q^i(t,\omega)$ in a manifold M (ω is the random process label) with a probability density $\rho(q,t)$ and satisfying a deterministic equation (**2B**) $\dot{q}^i(t,\omega) = (dq^i/dt)(t,\omega) = v^i(q(t,\omega),t)$ with random initial conditions $q^i(t_0,\omega) = q_0^i(\omega)$. The probability density will satisfy (**2C**) $\partial_t\rho + \partial_i(\rho v^i) = 0$ with initial data $\rho_0(q)$. Let $L(q,\dot{q},t)$ be some Lagrangian for the particle and define an equivalent Lagrangian via

$$L^*(q,\dot{q},t) = L(q,\dot{q},t) - \partial_t S + q^i\partial_i S \tag{2.1}$$

for some function S. The velocity field $v^i(q,t)$ yielding a classical motion with probability one can be found by minimizing the action functional

$$I(t_0,t_1) = E\left[\int_{t_0}^{t_1} L^*(q(t,\omega),\dot{q}(t,\omega),t)dt\right] \tag{2.2}$$

This leads to (**2D**) $\partial_t S + H(q,\nabla S,t) = 0$ and $p_i = (\partial L/\partial\dot{q}^i) = \partial_t S$ where $H \sim p_i\dot{q}^i - L$ with $v^i(q,t) = (\partial H/\partial p_i)(q,\nabla S(q,t),t)$. Now suppose that some geometric structure is given on M via $ds^2 = g_{ij}dq^idq^j$ so that a scalar curvature $\mathcal{R}(q,t)$ is meaningful and write the acutal Lagrangian as (**2E**) $L = L_C + \gamma(\hbar^2/m)\mathcal{R}(q,t)$ where γ will turn out to have the form $\gamma = (1/8)[(n-2)/(n-1)] = 1/16$ for $n=3$. Assume that in a transplantation $q^i \to q^i + dq^i$ the length of a vector $\ell = (g_{ik}A^iA^k)^{1/2}$ varies according to the law (**2F**) $\delta\ell = \ell\phi_k dq^k$ where the ϕ_k are covariant components of an arbitrary vector of M (this characterizes a Weyl geometry). One imagines that physics determines geometry so that the ϕ_k must be determined from some averaged least action principle yielding the motion of the particle; in particular the minimum now in (2.2) is to be evaluated with respect to the class of all Weyl geometries with fixed metric tensor. Since the only term containing the gauge vector $\vec{\phi} = (\phi_k)$ is the curvature term one requires

$E[\mathcal{R}(q(t,\omega)t] = minimum$ ($\gamma > 0$ for $n \geq 3$). This minimization yields

$$\mathcal{R} = \dot{\mathcal{R}} + (n-1)\left[(n-2)\phi_i\phi^i - 2\left(\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\phi^i)\right)\right] \tag{2.3}$$

where $\phi^i = g^{ik}\phi_k$ and $\dot{\mathcal{R}}$ is the Riemannian curvature based on the metric. Note here that a Weyl geometry is assumed as the proper background for the motion. One shows that the quantity $\hat{\rho}(q,t) = \rho(q,t)/\sqrt{g}$ transforms as a scalar under coordinate changes and a covariant equation of the form (**2G**) $\partial_t\hat{\rho} + (1/\sqrt{g})\partial_i(\sqrt{g}v^i\hat{\rho}) = 0$ ensues (g_{ik} is assumed time independent). Some calculation gives then a minimum when (**2H**) $\phi_i(q,t) = -[1/(n-2)]\partial_i log(\hat{\rho})]$. This shows that the transplantation properties of space are determined by the presence of matter and in turn this change in geometry acts on the particle via a "quantum" force $f_i = \gamma(\hbar^2/m)\partial_i\mathcal{R}$ depending on the gauge vector $\vec{\phi}$. Putting this $\vec{\phi}$ in (2.3) yields

$$R_w = \mathcal{R} = \dot{\mathcal{R}} + \frac{1}{2\gamma\sqrt{\hat{\rho}}}\left[\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\hat{\rho}}\right] \tag{2.4}$$

along with a (HJ) equation

$$\partial_t S + H_C(q, \nabla S, t) - \gamma\left(\frac{\hbar^2}{m}\right)\mathcal{R} = 0 \tag{2.5}$$

and for certain Hamiltonians of the form (**2I**) $H_C = (1/2m)g^{ik}(p_i - A_i)(p_k - A_k) + V$ with arbitrary fields A_k and V it is shown that the function $\psi = \sqrt{\hat{\rho}}exp[(i/\hbar)S(q,t)]$ satisfies a SE (omitting the A_i)

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\left[\partial_i\left(\sqrt{g}g^{ik}\partial_k\right)\right]\psi + \left[V - \gamma\left(\frac{\hbar^2}{m}\right)\dot{\mathcal{R}}\right]\psi \tag{2.6}$$

This Hamiltonian is characteristic of a particle in an EM field and all Hamiltonians arising in nonrelativistic applications may be reduced to the above form with corresponding HJ equation

$$\partial_t S = \frac{1}{2m}g^{ik}\partial_i S\partial_k S + V - \gamma\frac{\hbar^2}{m}\mathcal{R} = 0 \tag{2.7}$$

(note there are mistakes in the SE in [102] and in the improperly corrected form of [21]).

REMARK 2.1. Note that indices are lowered or raised via use of g_{ij} or its inverse g^{ij}. The most complete sources of notation for differential calculus on Riemannian manifolds seem to be [1, 126]. It is seen that $\hbar$ arises only via (2.5) and for $\dot{\mathcal{R}} = 0$ there is no $\hbar$ in the SE. If $\mathcal{R} = 0$ the quantum force is zero and "quantum mechanics" involves no $\hbar$; $\mathcal{R} = 0$ (with $\dot{\mathcal{R}} = 0$) involves (2.10) below giving $Q = 0$. ■

Now given (2.7), and comparing to (1.1) for example, we see that (**2J**) $Q \sim -\gamma(\hbar^2/m)\mathcal{R}$ with $\mathcal{R}$ given by (2.4) and $\gamma = 1/16$ for $n = 3$. Thus

$$Q \sim -\frac{\hbar^2}{16m}\left[\dot{\mathcal{R}} + \frac{8}{\sqrt{\hat{\rho}g}}\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\hat{\rho}})\right] \tag{2.8}$$

and the SE (2.6) contains only $\dot{\mathcal{R}}$. Further from (**2H**) we have for the Weyl vector $\phi_i = -\partial_i log(\hat{\rho}) = -\partial_i\hat{\rho}/\hat{\rho}$ and there is an expression for $\mathcal{R}$ in the form (2.3) leading to

$$Q \sim -\frac{\hbar^2}{16m}\left[\dot{\mathcal{R}} + 2\left\{\phi_i\phi^i - \frac{2}{\sqrt{g}}\partial_i(\sqrt{g}\phi^i)\right\}\right] \tag{2.9}$$

showing how Q depends directly on the Weyl vector. When $\dot{\mathcal{R}} = 0$ (flat space) one sees that the SE is classical and (**2K**) $Q = -(\hbar^2/8m)[\phi_i\phi^i - (1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)]$. Note that when $g = 1$ (so $\dot{\mathcal{R}} = 0$ automatically) and $\hat{\rho} = \rho$ we have then

$$\phi_k\phi^k - 2\partial_k\phi^k \sim -\left(\frac{|\nabla\rho|^2}{\rho^2} - \frac{2\Delta\rho}{\rho}\right) = 4\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{2.10}$$

which means (**2L**) $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$ as in the desired (**1C**).

REMARK 2.2 Thus starting with a manifold M with metric g_{ij} and random initial conditions as indicated for a particle of mass m, the resulting classical statistical dynamics based on a probability distribution P with $\rho = mP$ can be properly phrased in a Weyl geometry in which the particle undergoes classical motion with probability one. The assumed Weyl geometry as well as the particle motion is determined via $\hat{\rho}(\rho, g)$ which says that given a different P there will be a different ρ and $\hat{\rho}$ (since g is fixed). Hence writing $\psi = \sqrt{\hat{\rho}}exp(iS/\hbar)$ one expects a different quantum potential and a different Weyl geometry. The SE will however remain unchanged and this may be a solution to the apparent problems illustrated in Section 1 about different quantum potentials being attached to the same SE. Another point of view could be that for m fixed each P (or equivalently ρ) determines a P-dependent motion via it Weyl geometry and each such motion can be described by a P-dependent wave function. The choice of $\hbar$ is arbitrary; here it arises via ψ and any $\hbar$ will do. The identification with Planck's constant has to come from other considerations. ∎

We go now to the second paper in [102] and sketch an interesting role of Weyl geometry in the Klein-Gordon (KG) equation (cf. also [21, 29, 24, 33] for discussion of this approach). The idea is to start from first principles, extended to gauge invariance relative to an arbitrary choice of spacetime calibration. Weyl geometry is not assumed but derived with the particle motion from a single average action principle. Thus assume a generic 4-D

manifold with torsion free connection $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$ and a metric tensor g with signature $(+,-,-,-)$; $\hbar = c = 1$ is taken for convenience (although this loses important information in the equations). The analysis will produce an integrable Weyl geometry with weights $w(g_{\mu\nu}) = 1$ and $w(\Gamma^{\lambda}_{\mu\nu}) = 0$ (cf. [21, 72] for Weyl geometry and Weyl-Dirac theory). One takes random initial conditions on a spacelike 3-D hypersurface and produces both particle motion and spacetime geometry via an average stationary action principle (**2M**) $\delta\left[E\int_{\tau_1}^{\tau_2} L(x(\tau),\dot{x}(\tau))d\tau\right] = 0$ where τ is an arbitrary parameter along the particle trajectory. Given L positively homogeneous of first degree in $\dot{x}^{\mu} = dx^{\mu}/d\tau$ and transforming as a scalar of weight $w(L) = 0$ as well as a gauge invariant probability measure it follows that the action integral will be parameter invariant, coordinate invariant, and gauge invariant. A suitable Lagrangian is (**2N**) $L(x,dx) = (m^2 - (1/6)\mathcal{R})^{1/2}ds + A_{\mu}dx^{\mu}$ where $ds = (g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu})^{1/2}d\tau$ and $w(m) = -1/2$ (m = rest mass corresponds to a scalar Weyl field with no equation needed and the factor (1/6) in L is for convenience later). One writes (**2O**) $A_{\mu} = \bar{A}_{\mu} - \partial_{\mu}S$ where $\bar{A}_{\mu} \sim$ EM 4-potential in Lorentz gauge and $w(S) = w(\bar{A}_{\mu}) = 0$.

Omitting here the considerable details of calculation (which are given in [102] and sketched in [21, 33]) one can work with a modified Lagrangian (**2P**) $\bar{L}(x,dx) = (m^2 - (1/6)\mathcal{R})^{1/2} + \bar{A}_{\mu}dx^{\mu}$. Variational methods lead to a 1-parameter family of hypersurfaces $S(x) = constant$ satisfying the HJ equation

$$g^{\mu\nu}(\partial_{\mu}S - \bar{A}_{\mu})(\partial_{\nu}S - \bar{A}_{\nu}) = m^2 - (1/6)\mathcal{R} \tag{2.11}$$

and a congruence of curves intersecting this family given via

$$\frac{dx^{\mu}}{ds} = \frac{g^{\mu\nu}(\partial_{\nu}S - \bar{A}_{\nu})}{[g^{\rho\sigma}(\partial_{\rho} - \bar{A}_{\rho})(\partial_{\sigma}S - \bar{A}_{\sigma})]^{1/2}} \tag{2.12}$$

The probability measure is determined by its probability current density j^{μ} where $\partial_{\mu}j^{\mu} = 0$ and (**2Q**) $j^{\mu} = \rho(\sqrt{-g}g^{\mu\nu}(\partial_{\nu}S - \bar{A}_{\nu})$. Gauge invariance implies $w(j^{\mu}) = 0 = w(S)$ and $w(\rho) = -1$ so ρ is the scalar probability density of the particle random motion. To find the connection the variational principle for (**2M**) is rephrased as

$$\delta\left[\int_{\Omega} d^4x[(m^2 - (1/6)\mathcal{R})(g_{\mu\nu}j^{\mu}j^{\nu})]^{1/2} + A_{\mu}j^{\mu}\right] = 0 \tag{2.13}$$

Since the $\Gamma^{\lambda}_{\mu\nu}$ arise only in $\mathcal{R}$ this reduces to (**2R**) $\delta[\int_{\Omega}\rho\mathcal{R}\sqrt{-g}d^4x] = 0$ where (2.11) has been used. This leads to

$$\Gamma^{\lambda}_{\mu\nu} = \left\{\begin{array}{c}\lambda\\ \mu\nu\end{array}\right\} + \frac{1}{2}(\phi_{\mu}\delta^{\lambda}_{\nu} + \phi_{\nu}\delta^{\lambda}_{\mu} - g_{\mu\nu}g^{\lambda\rho}\phi_{\rho});\ \ \phi_{\mu} = \partial_{\mu}log(\rho) \tag{2.14}$$

and shows that the connections are integrable Weyl connections with a gauge field ϕ_μ ((**2A**) suggests here perhaps $\phi_i = -(1/2)\partial_i log(\rho)$). The HJ equation (2.11) and $\partial_\mu j^\mu = 0$ can be combined into a single equation for $S(x)$, namely

$$e^{iS}g^{\mu\nu}(iD_\mu - \bar{A}_\mu)(iD_\nu - \bar{A}_\nu)e^{-iS} - (m^2 - (1/6)\mathcal{R}) = 0 \tag{2.15}$$

with $D_\mu\rho = 0$ where (cf. [3, 21])

$$D_\mu T^\alpha_\beta = \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\mu\epsilon}T^\epsilon_\beta - \Gamma^\epsilon_{\mu\beta}T^\alpha_\epsilon + w(T)\phi_\mu T^\alpha_\beta \tag{2.16}$$

(D_μ is called the double-covariant Weyl derivative and one notes that it is ρ and not m, as in [3], which behaves as a constant under D_μ). Then to any solution (ρ, S) of these equations corresponds a particular random motion for the particle. One notes that (2.15)-(2.16) can be written in a familiar KG form

$$\left(\frac{i}{\sqrt{-g}}\partial_\mu\sqrt{-g} - \bar{A}_\mu\right)g^{\mu\nu}(i\partial_\nu - \bar{A}_\nu)\psi - (m^2 - (1/6)\dot{\mathcal{R}})\psi = 0 \tag{2.17}$$

where $\psi = \sqrt{\rho}exp(-iS)$ and $\dot{\mathcal{R}}$ is the Riemannian scalar curvature. We have also from [102]

$$\mathcal{R} = \dot{\mathcal{R}} - 3\left[\frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu + \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\phi_\nu\right] = \dot{\mathcal{R}} + \mathcal{R}_w \tag{2.18}$$

in keeping also with [33].

REMARK 2.3. Note here $g^{\mu\nu}\phi_\nu = \phi^\mu$ so (2.18) gives for the last term (**2S**) $\mathcal{R}_w = -3[(1/2)\phi_\mu\phi^\mu + (1/\sqrt{-g})\partial_\mu(\sqrt{-g}\phi^\mu)]$ whereas (2.3) suggests here (★) $-3[2\phi_\mu\phi^\mu - (2/\sqrt{-g})\partial_\mu(\sqrt{-g}\phi^\mu)]$ which is similar to paper 3 of [102] in having a minus sign in the middle; we remark that a change $\phi_\mu \to -2\phi_\mu$ would produce some agreement and will stay with (2.18) or equivalently (**2S**) due to calculations in Remark 2.5. ■

REMARK 2.4. We add here a few standard formulas involving derivatives; thus

$$\nabla_\mu\lambda^\nu = \partial_\mu\lambda^\nu + \Gamma^\mu_{\rho\nu}\lambda^\rho;\ \nabla_\mu\lambda^\mu = \partial_\mu\lambda^\mu + \Gamma^\mu_{\rho\mu}\lambda^\rho\ (divergence); \tag{2.19}$$

$$\nabla_\mu\lambda_\nu = \partial_\mu - \Gamma^\rho_{\nu\mu}\lambda_\rho;\ \Gamma^\mu_{\rho\mu} = \partial_\rho log(\sqrt{g});\ \nabla_m\lambda^m = \frac{1}{\sqrt{g}}\partial_m(\sqrt{g}\lambda^m)$$

Also from (2.17)

$$\Box \sim \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) = \nabla_\mu g^{\mu\nu}\partial_\nu = \nabla_\mu\nabla^\mu \tag{2.20}$$

since $\nabla^\mu \sim \partial^\mu$ acting on functions (one could use $|g|$ instead of $\pm g$). ■

REMARK 2.5. We see that for $\bar{A}_\mu = 0$ the HJ equation (2.11) has the form (**2T**) $\partial_\mu S\partial^\mu S = m^2 - (1/6)\mathcal{R}$ and mention that it is shown in

[33] that the 1/6 factor is essential if one wants a linear KG equation. We want now to identify $\mathcal{R}_w$ with a multiple of Q which should have a form like (**2U**) $Q \propto (1/\sqrt{\rho})\nabla^\mu\nabla_\mu(\sqrt{\rho})$. A crude calculation suggests
(2.21)

$$\partial_\mu\partial^\mu\sqrt{\rho} = \partial_\mu\left[\frac{1}{2}\rho^{-1/2}\partial^\mu\rho\right] = \frac{1}{2}\left[-\frac{1}{2}\rho^{-3/2}\partial_\mu\rho\partial^\mu\rho + \rho^{-1/2}\partial_\mu\partial^\mu\rho\right] \Rightarrow$$

$$\Rightarrow \frac{\partial_\mu\partial^\mu\sqrt{\rho}}{\sqrt{\rho}} = \frac{1}{2}\left[-\frac{1}{2}\frac{\partial_\mu\rho\partial^\mu\rho}{\rho^2} + \frac{\partial_\mu\partial^\mu\rho}{\rho}\right]$$

and it is easy to check (cf. [88]) that $\nabla_m(fg^m) = (\nabla_m f)g^m + f(\nabla_m g^m)$. Hence $\nabla_m\nabla^m\sqrt{\rho}$ can be written out as in (2.21) to get

$$\frac{\Box(\sqrt{\rho})}{\sqrt{\rho}} = \frac{1}{2}\left[-\frac{1}{2}\frac{\nabla_\mu\rho\nabla^\mu\rho}{\rho^2} + \frac{\Box(\rho)}{\rho}\right] \tag{2.22}$$

and hence from (2.18)

$$\mathcal{R}_w = -3\left[\frac{1}{2}\frac{\nabla_\mu\rho\nabla^\mu\rho}{\rho^2} + \nabla_\mu\left(\frac{\nabla^\mu\rho}{\rho}\right)\right] = -3\left[\frac{1}{2}\frac{\nabla_\mu\rho\nabla^\mu\rho}{\rho^2} + \right. \tag{2.23}$$

$$\left. + \frac{\nabla_\mu\nabla^\mu\rho}{\rho} - \frac{\nabla_\mu\rho\nabla^\mu\rho}{\rho^2}\right] = -6\frac{\Box(\sqrt{\rho})}{\sqrt{\rho}}$$

The formula for Q is then (**2V**) $Q = -[\Box(\sqrt{\rho})/\sqrt{\rho}] = (1/6)\mathcal{R}_w$. We remark that in various contexts formulas for Q arise here with multipliers $1/m^2$, $\hbar^2/2m$, etc. (cf. [21] and remarks below). ■

3. EMERGENCE OF Q IN GEOMETRY

In [21] we have indicated a number of contexts where Q arises in geometrical situations involving KG type equations and we review this here (cf. also [29]). We list a number of occasions (while omitting others).

(1) We omit any details for the Bertoldi-Faraggi-Matone (BFM) approach (see [9, 21, 45, 46, 47]) since it involves a whole philosophy (of considerable importance). Thus for $\eta^{\mu\nu} = diag(-1,1,1,1)$ and $q = (ct, q_1, q_2, q_3)$ one has

$$\frac{1}{2m}\eta^{\mu\nu}\partial_\mu S^{cl}\partial_\nu S^{cl} + \mathfrak{W}'_{rel} = 0; \tag{3.1}$$

with $\mathfrak{W}'_{rel} = \frac{1}{2mc^2}[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S^{cl}]$ where V is some potential which we could take to be zero. The quantum version attaches Q to (3.1) to get ($S^{cl} \to S$)

$$\frac{1}{2m}(\partial S)^2 + \mathfrak{W}_{rel} + Q = 0;\ \mathfrak{W}_{rel} = \frac{1}{2mc^2}[m^2c^4 - V^2 - 2cV\partial_0 S] \tag{3.2}$$

This involves then

$$\mathfrak{W}_{rel} = \left(\frac{\hbar^2}{2m}\right)\frac{\Box(Re^{iS/\hbar})}{Re^{iS/\hbar}};\ Q = -\frac{\hbar^2}{2m}\frac{\Box R}{R};\ \partial\cdot(R^2\partial S) = 0 \tag{3.3}$$

where one uses $\partial \sim \nabla$ when $g_{\mu\nu} = \eta_{\mu\nu}$.

(2) One can derive the SE, the KG equation, and the Dirac equation using methods of scale relativity (cf. [21, 29, 35, 38, 86, 87, 94]); here e.g. quantum paths are considered to be continuous nondifferentiable curves with left and right derivatives at any point. Using a "diffusion" coefficient $D = \hbar/2m$ as in the Nelson theory (cf. [21, 29, 82]) one defines "average" velocities $V = (1/2)[d_+x(t) + d_-x(t)]$ and $U = (1/2)[d_+x(t) - d_-x(t)]$. Then e.g. there is a SE $i\hbar\psi_t = -(\hbar^2/2m)\Delta\psi + \mathfrak{U}\psi$ with quantum potential $Q = -(m/2)U^2 - (\hbar/2)\partial U$ where $U = (\hbar/m)(\partial\sqrt{\rho}/\sqrt{\rho})$. The ideas should be extendible to a KG equation where $Q \sim (\hbar^2/m^2c^2)(\Box_g|\psi|/\|psi|)$ (see Section 5).

(3) One can construct directly a KG theory following [83] in the form $(\partial_0^2 - \nabla^2 + m^2)\phi = 0$ where $\eta_{\mu\nu} = (1,-1,-1,-1)$. If $\psi = \phi^+$ with $\psi^* = \phi^-$ correspond to positive and negative frequency parts of $\phi = \phi^+ + \phi^-$ the particle current is $j_\mu = i\psi^*\overleftrightarrow{\partial}_\mu\psi$ and $N = \int d^3x j_0$ is the particle number. Trajectories have the form $d\mathbf{x}/dt = \mathbf{j}(t,\mathbf{x})/j_0(t,\mathbf{x})$ for $t = x_0$ and for $c = \hbar = 1$ one arrives at

$$\partial^\mu(R^2\partial_\mu S) = 0;\ \frac{(\partial^\mu S)(\partial_\mu S)}{2m} - \frac{m}{2} + Q = 0;\ Q = -\frac{1}{2m}\frac{\partial^\mu\partial_\mu R}{R} \tag{3.4}$$

(4) A covariant field theoretic version is also given in [83] using deDonder-Weyl theory (cf. also [21, 29]). One works with a real scalar field $\phi(x)$ and defines (**3A**) $\mathfrak{A} = \int d^4x\mathfrak{L};\ \mathfrak{L} = (1/2)(\partial^\mu\phi)(\partial_\mu\phi) - V(\phi)$ with (**3B**) $\pi^\mu = \partial\mathfrak{L}/\partial(\partial_\mu\phi) = \partial^\mu\phi,\ \partial_\mu\phi = \partial\mathfrak{H}/\partial\pi^\mu$, and $\partial_\mu\pi^\mu = -\partial\mathfrak{H}/\partial\phi$. One takes a preferred foliation of spacetime with R^μ normal to the leaf Σ and writes $\mathfrak{R}([\phi],\Sigma) = \int_\Sigma d\Sigma_\mu R^\mu$ with $\mathfrak{S}([\phi],\Sigma) = \int_\Sigma d\Sigma_\mu S^\mu$ and $\Psi = \mathfrak{R}exp(i\mathfrak{S}/\hbar)$. A covariant version of Bohmian mechanics ensues with

$$\frac{1}{2}\frac{dS_\mu}{d\phi}\frac{dS^\mu}{d\phi} + V + Q + \partial_\mu S^\mu = 0;\ \frac{dR^\mu}{d\phi}\frac{dS^\mu}{d\phi} + J + \partial_\mu R^\mu = 0 \tag{3.5}$$

$$Q = -\frac{\hbar^2}{2\mathfrak{R}}\frac{\delta^2\mathfrak{R}}{\delta_\Sigma\phi^2};\ J = \frac{\mathfrak{R}}{2}\frac{\delta^2\mathfrak{S}}{\delta_\Sigma\phi^2} \tag{3.6}$$

The nature of this approach as a covariant version of the Bohmian hidden variable theory is spelled out in the last paper of [83]. This is a significant extension of earlier classical field theoretic approaches and another lovely extension is described by Nikolic in [84] involving

a covariant many fingered time Bohmian interpretation of quantum field theory (QFT).

We preface the next set of examples with a discussion of a formula $\mathfrak{M}^2 = m^2 exp(\mathfrak{Q}_{rel})$ used in [105] in an important manner and produced also in [85]. This formula differs from the result $\mathfrak{M} = mexp(\mathfrak{Q}_{rel})$ of [108] (which was abandoned in [105]) and in order to clarify this we write out in more detail the approach of [85]. Thus one is dealing with a Bohmian theory and for a Klein-Gordon (KG) equation a wave function $\psi = Rexp(iS/\hbar)$ this leads to

$$\partial_\mu(R^2\partial^\mu S) = 0;\ \partial_\mu S\partial^\mu S = \mathfrak{M}^2c^2\ (\sim m^2c^2(1+\mathfrak{Q}_{rel})) \tag{3.7}$$

where $\mathfrak{Q}_{rel} = (\hbar^2/m^2c^2)(\partial_\mu\partial^\mu R/R)$ and (temporarily now) $\partial_\mu\partial^\mu \sim \Box = (1/c^2)\partial_t^2 - \Delta$ where $\eta_{\alpha\beta} = diag(1,-1,-1,-1)$. Now $\mathfrak{M} = 1+\mathfrak{Q}_{rel}$ is only an approximation (leading e.g. to tachyon problems) and a better formula for $\mathfrak{M}$ can be found as follows. Thus one knows (**3C**) $(dx^\mu(\tau)/d\tau) = (1/m)\partial^\mu S$ and differentiating gives

$$\partial_\tau\partial^\mu S = \partial_\nu\partial^\mu S\frac{dx^\mu}{d\tau} = \partial_\nu\mathfrak{M}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + \mathfrak{M}\frac{d^2x^\mu}{d\tau^2} \tag{3.8}$$

But via the formula (valid for $g_{ab} = \eta_{ab}$ constant

$$\partial_b(\partial_a S\partial^a S) = (\partial_b\partial_a S)(\partial^a S) + (\partial_a S)(\partial_b\partial^a S); \tag{3.9}$$

$$\partial_a\partial_b\partial^a S = \partial_a S\eta^{ac}\partial_c\partial_b S = \partial^c S\partial_c\partial_b S$$

one has (**3D**) $\partial_\nu(\partial_\mu S\partial^\mu S) = 2(\partial^\mu S)(\partial_\mu\partial_\nu S)$ and therefore

$$\partial_\nu(\partial_\mu S\partial^\mu S) = \partial_\nu(\mathfrak{M}^2c^2) = 2\mathfrak{M}\partial_\nu\mathfrak{M}c^2 = 2(\partial^\nu S)(\partial_\mu\partial_\nu S) = \tag{3.10}$$

$$= 2\mathfrak{M}\frac{dx^\mu}{d\tau}(\partial_\mu\partial_\nu S)$$

Hence (**3E**) $\partial_\nu\mathfrak{M}c^2 = (\partial_\mu\partial_\nu S)(dx^\mu/d\tau)$ which implies

$$\eta^{\alpha\nu}c^2\partial_\nu\mathfrak{M} = \eta^{\alpha\nu}\partial_\mu\partial_\nu S(dx^\mu/d\tau) = \partial_\mu\partial^\alpha S(dx^\mu/d\tau) \tag{3.11}$$

Consequently (8.2) becomes

$$\eta^{\alpha\nu}c^2\partial_\nu\mathfrak{M} = \partial_\nu\mathfrak{M}\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau} + \mathfrak{M}\frac{d^2x^\alpha}{d\tau^2} \equiv \tag{3.12}$$

$$\equiv \mathfrak{M}\frac{d^2x^\alpha}{d\tau^2} = \left(c^2\eta^{\alpha\nu} - \frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau}\right)\partial_\nu\mathfrak{M}$$

and this is equation (9) of [85]. For $|\dot{x}^\alpha| << c$ one obtains then $\mathfrak{M}\ddot{x}^\alpha \sim c^2\partial^\alpha\mathfrak{M} \sim -c^2\partial_\alpha\mathfrak{M}$ and comparing with the nonrelativistic equation $m\ddot{x}^\alpha = -\partial_\alpha Q_{cl}$ implies $\mathfrak{M} \sim mexp(\mathfrak{Q}_{cl}/mc^2)$ and suggests that $\mathfrak{M} \sim mexp(\mathfrak{Q}_{rel}/2)$ (recall $\mathfrak{Q}_{cl} = -(\hbar^2/2m)(\nabla^2|\psi|/|\psi|)$).

Now one observes that the quantum effects will affect the geometry and in fact are equivalent to a change of spacetime metric

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = (\mathfrak{M}^2/m^2)g_{\mu\nu} \tag{3.13}$$

(conformal transformation). The QHJE becomes $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu S\tilde{\nabla}_\nu S = m^2c^2$ where $\tilde{\nabla}_\mu$ represents covariant differentiation with respect to the metric $\tilde{g}_{\mu\nu}$ and the continuity equation is then $\tilde{g}_{\mu\nu}\tilde{\nabla}_\mu(\rho\tilde{\nabla}_\nu S) = 0$. The important conclusion here is that the presence of the quantum potential is equivalent to a curved spacetime with its metric given by (3.13). This is a geometrization of the quantum aspects of matter and it seems that there is a dual aspect to the role of geometry in physics. The spacetime geometry sometimes looks like "gravity" and sometimes reveals quantum behavior. The curvature due to the quantum potential may have a large influence on the classical contribution to the curvature of spacetime. The particle trajectory can now be derived from the guidance relation via differentiation as in (**3C**) again, leading to the Newton equations of motion

$$\mathfrak{M}\frac{d^2x^\mu}{d\tau^2} + \mathfrak{M}\Gamma^\mu_{\nu\kappa}u^\nu u^\kappa = (c^2g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu\mathfrak{M} \tag{3.14}$$

Using the conformal transformation above (8.7) reduces to the standard geodesic equation.

We extract now from [21, 105, 106, 107, 108, 109, 110, 111, 112, 113] with emphasis on the survey article [105]. This may seem overly repetitious but the material seems worthy of further emphasis. Thus a general "canonical" relativistic system consisting of gravity and classical matter (no quantum effects) is determined by the action

$$\mathcal{A} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\mathcal{R} + \int d^4x\sqrt{-g}\frac{\hbar^2}{2m}\left(\frac{\rho}{\hbar^2}\mathcal{D}_\mu S\mathcal{D}^\mu S - \frac{m^2}{\hbar^2}\rho\right) \tag{3.15}$$

where $\kappa = 8\pi G$ and $c = 1$ for convenience and $\mathcal{D}_\mu$ is the covariant derivative based on $g_{\mu\nu}$ ($\mathcal{D}_\mu \sim \nabla_\mu$). It was seen above that via deBroglie the introduction of a quantum potential is equivalent to introducing a conformal factor $\Omega^2 = \mathfrak{M}^2/m^2$ in the metric. Hence in order to introduce quantum effects of matter into the action (3.15) one uses this conformal transformation to get ($1 + Q \sim exp(Q)$ and $Q \sim (\hbar^2/c^2m^2)(\Box(\sqrt{\rho})/\sqrt{\rho})$ with $c = 1$ here)

$$\begin{aligned}\mathfrak{A} = {} & \frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}(\bar{\mathcal{R}}\Omega^2 - 6\bar{\nabla}_\mu\Omega\bar{\nabla}^\mu\Omega) + \\ & + \int d^4x\sqrt{-\bar{g}}\left(\frac{\rho}{m}\Omega^2\bar{\nabla}_\mu S\bar{\nabla}^\mu S - m\rho\Omega^4\right) + \\ & + \int d^4x\sqrt{-\bar{g}}\lambda\left[\Omega^2 - \left(1 + \frac{\hbar^2}{m^2}\frac{\bar{\Box}\sqrt{\rho}}{\sqrt{\rho}}\right)\right]\end{aligned} \tag{3.16}$$

where a bar over any quantity means that it corresponds to the nonquantum regime. Here only the first two terms of the expansion of $\mathfrak{M}^2 = m^2 exp(\mathfrak{Q})$ have been used, namely $\mathfrak{M}^2 \sim m^2(1+\mathfrak{Q})$. λ is a Lagrange multiplier introduced to identify the conformal factor with its Bohmian value. One uses here $\bar{g}_{\mu\nu}$ to raise or lower indices and to evaluate the covariant derivatives; the physical metric (containing the quantum effects of matter) is $g_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu}$. By variation of the action with respect to $\bar{g}_{\mu\nu}$, Ω, ρ, S, and λ one arrives at quantum equations of motion, including quantum Einstein equations (cf. [21, 105]). There is a generalized equivalence principle. The gravitational effects determine the causal structure of spacetime as long as quantum effects give its conformal structure. This does not mean that quantum effects have nothing to do with the causal structure; they can act on the causal structure through back reaction terms appearing in the metric field equations. The conformal factor of the metric is a function of the quantum potential and the mass of a relativistic particle is a field produced by quantum corrections to the classical mass. One has shown that the presence of the quantum potential is equivalent to a conformal mapping of the metric. Thus in different conformally related frames one "feels" different quantum masses and different curvatures. In particular there are two frames with one containing the quantum mass field and the classical metric while the other contains the classical mass and the quantum metric. In general frames both the spacetime metric and the mass field have quantum properties so one can state that different conformal frames are identical pictures of the gravitational and quantum phenomena. One "feels" different quantum forces in different conformal frames. The question then arises of whether the geometrization of quantum effects implies conformal invariance just as gravitational effects imply general coordinate invariance. One sees here that Weyl geometry provides additional degrees of freedom which can be identified with quantum effects and seems to create a unified geometric framework for understanding both gravitational and quantum forces. Some features here are: (i) Quantum effects appear independent of any preferred length scale. (ii) The quantum mass of a particle is a field. (iii) The gravitational constant is also a field depending on the matter distribution via the quantum potential. (iv) A local variation of matter field distribution changes the quantum potential acting on the geometry and alters it globally; the nonlocal character is forced by the quantum potential (cf. [21, 105, 112]).

Next (still following [105]) one goes to Weyl geometry based on the Weyl-Dirac action

$$\mathfrak{A} = \int d^4x\sqrt{-g}(F_{\mu\nu}F^{\mu\nu} - \beta^2\ {}^W\mathcal{R} + (\sigma+6)\beta_{;\mu}\beta^{;\mu} + \mathfrak{L}_{matter}) \tag{3.17}$$

Here $F_{\mu\nu}$ is the curl of the Weyl 4-vector ϕ_μ, σ is an arbitrary constant and β is a scalar field of weight -1. The symbol ";" represents a covariant derivative under general coordinate and conformal transformations (Weyl covariant derivative) defined as $X_{;\mu} = {}^W\nabla_\mu X - \mathcal{N}\phi_\mu X$ where $\mathcal{N}$ is the Weyl weight of X. The equations of motion are then given in [21, 105]. There is then agreement with the Bohmian theory provided one identifies

$$\beta \sim \mathfrak{M};\ \frac{8\pi\mathfrak{T}}{\mathcal{R}} \sim m^2;\ \frac{1}{\sigma\phi_\alpha\phi^\alpha - (\mathcal{R}/6)} \sim \alpha = \frac{\hbar^2}{m^2c^2} \tag{3.18}$$

Thus β is the Bohmian quantum mass field and the coupling constant α (which depends on $\hbar$) is also a field, related to geometrical properties of spacetime. One notes that the quantum effects and the length scale of the spacetime are related. To see this suppose one is in a gauge in which the Dirac field is constant; apply a gauge transformation to change this to a general spacetime dependent function, i.e.

$$\beta = \beta_0 \to \beta(x) = \beta_0 exp(-\Xi(x));\ \phi_\mu \to \phi_\mu + \partial_\mu\Xi \tag{3.19}$$

Thus the gauge in which the quantum mass is constant (and the quantum force is zero) and the gauge in which the quantum mass is spacetime dependent are related to one another via a scale change. In particular ϕ_μ in the two gauges differ by $-\nabla_\mu(\beta/\beta_0)$ and since ϕ_μ is a part of Weyl geometry and the Dirac field represents the quantum mass one concludes that the quantum effects are geometrized which shows that ϕ_μ is not independent of β so the Weyl vector is determined by the quantum mass and thus the geometrical aspects of the manifold are related to quantum effects).

3.1. **QUANTUM POTENTIAL AS A DYNAMICAL FIELD.** In [105, 112] (cf. also [21]) one can write down a scalar tensor theory where the conformal factor and the quantum potential are both dynamical fields but first we deal with (3.16). For the relativistic situation one will have e.g. $\mathfrak{Q} = (\hbar^2/m^2c^2)(\Box_g|\psi|/|\psi|)$ where $\Box_g|\psi| \sim \nabla_\alpha\nabla^\alpha|\psi| = g^{\alpha\beta}\nabla_\beta\nabla_\alpha|\psi|$ and the HJ equation is $\nabla_\mu S\nabla^\mu S = \mathfrak{M}^2c^2$ where $\mathfrak{M}^2 = m^2exp(\mathfrak{Q})$. Equivalently $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu S\tilde{\nabla}_\nu S = m^2c^2$ where $g_{\mu\nu} = (\mathfrak{M}/m)^2\tilde{g}_{\mu\nu}$ and $\tilde{\nabla}_\mu$ is the covariant derivative with respect to $\tilde{g}_{\mu\nu}$. The corresponding geodesic equation is given via (3.14). We write $\Omega^2 = (\mathfrak{M}/m)^2$ and this leads to (3.16) based on the fundamental action (3.15). Recall here $exp(\mathfrak{Q}) \sim m^2(1+\mathfrak{Q})$ has been used for $\mathfrak{M}$ in the last term in (3.16). We recall also the fundamental equations determined by varying the action (3.16) with respect to $\bar{g}_{\mu\nu}$, Ω, ρ, S, and λ are (cf. [21, 105])

(1) The equation of motion for Ω

$$\bar{\mathcal{R}}\Omega + 6\,\bar{\Box}\,\Omega + \frac{2\kappa}{m}\rho\Omega(\bar{\nabla}_\mu S\bar{\nabla}^\mu S - 2m^2\Omega^2) + 2\kappa\lambda\Omega = 0 \tag{3.20}$$

(2) The continuity equation for particles $\bar{\nabla}_\mu(\rho\Omega^2\bar{\nabla}^\mu S) = 0$

(3) The equations of motion for particles

$$(\bar{\nabla}_\mu S\bar{\nabla}^\mu S - m^2\Omega^2)\Omega^2\sqrt{\rho} + \frac{\hbar^2}{2m}\left[\bar{\Box}\left(\frac{\lambda}{\sqrt{\rho}}\right) - \lambda\frac{\bar{\Box}\sqrt{\rho}}{\rho}\right] = 0 \tag{3.21}$$

(4) The modified Einstein equations for $\bar{g}_{\mu\nu}$

(3.22)

$$\Omega^2\left[\bar{\mathcal{R}}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\mathcal{R}}\right] - [\bar{g}_{\mu\nu}\,\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu]\Omega^2 - 6\bar{\nabla}_\mu\Omega\bar{\nabla}_\nu\Omega + 3\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\Omega\bar{\nabla}^\alpha\Omega +$$
$$+\frac{2\kappa}{m}\rho\Omega^2\bar{\nabla}_\mu S\bar{\nabla}_\nu S - \frac{\kappa}{m}\rho\Omega^2\bar{g}_{\mu\nu}\bar{\nabla}_\alpha S\bar{\nabla}^\alpha S + \kappa m\rho\Omega^4\bar{g}_{\mu\nu} +$$
$$+\frac{\kappa\hbar^2}{m^2}\left[\bar{\nabla}_\mu\sqrt{\rho}\bar{\nabla}_\nu\left(\frac{\lambda}{\sqrt{\rho}}\right) + \bar{\nabla}_\nu\sqrt{\rho}\bar{\nabla}_\mu\left(\frac{\lambda}{\sqrt{\rho}}\right)\right] - \frac{\kappa\hbar^2}{m^2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\left[\lambda\frac{\bar{\nabla}^\alpha\sqrt{\rho}}{\sqrt{\rho}}\right] = 0$$

(5) The constraint equation $\Omega^2 = 1 + (\hbar^2/m^2)[(\bar{\Box}\sqrt{\rho})/\sqrt{\rho}]$

Thus the back reaction effects of the quantum factor on the background metric are contained in these highly coupled equations. A simpler form of (9.1) can be obtained by taking the trace of (9.2) and using (9.1) which produces $\lambda = (\hbar^2/m^2)\bar{\nabla}_\mu[\lambda(\bar{\nabla}^\mu\sqrt{\rho})/\sqrt{\rho}]$. A solution of this via perturbation methods using the small parameter $\alpha = \hbar^2/m^2$ yields the trivial solution $\lambda = 0$ so the above equations reduce to

$$\bar{\nabla}_\mu(\rho\Omega^2\bar{\nabla}^\mu S) = 0;\ \bar{\nabla}_\mu S\bar{\nabla}^\mu S = m^2\Omega^2;\ \mathfrak{G}_{\mu\nu} = -\kappa\mathfrak{T}^{(m)}_{\mu\nu} - \kappa\mathfrak{T}^{(\Omega)}_{\mu\nu} \tag{3.23}$$

where $\mathfrak{T}^{(m)}_{\mu\nu}$ is the matter energy-momentum (EM) tensor and

$$\kappa\mathfrak{T}^{(\Omega)}_{\mu\nu} = \frac{[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]\Omega^2}{\Omega^2} + 6\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\omega^2} - 2g_{\mu\nu}\frac{\nabla_\alpha\Omega\nabla^\alpha\Omega}{\Omega^2} \tag{3.24}$$

with $\Omega^2 = 1 + \alpha(\bar{\Box}\,\sqrt{\rho}/\sqrt{\rho})$. Note that the second relation in (9.4) is the Bohmian equation of motion and written in terms of $g_{\mu\nu}$ it becomes $\nabla_\mu S\nabla^\mu S = m^2c^2$. Many examples with a lot of expansion is to be found in [21, 105] and references there.

4. OTHER GEOMETRIC ASPECTS

The quantum potential arises in many geometrical and cosmological situations and we mention a few of these here.

(1) We have written about the Wheeler-deWitt (WDW) equation and the QP in [28] at some length and in [21] have discussed the QP in related geometric situations following [92, 93, 104, 105, 114] in particular. For background information on WDW we refer to [73] for example. One thinks of an ADM situation with (**4A**) $ds^2 = -(N^2 - h^{ij}H_iN_j)dt^2 + 2N_idx^idt + h_{ij}dx^idx^j$ and the deWitt metric (**4B**) $G_{ijk\ell} = (1/\sqrt{h})h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell})$. Given a wave

function $\psi = \sqrt{P}exp(iS/\hbar)$ where P can be thought of in terms of momentum fluctuations $(1/P)(\delta P/\delta h_{ij})$ one finds a quantum potential

$$Q = -\frac{\hbar^2}{2}P^{-1/2}\frac{\delta}{\delta h_{ij}}\left(G_{ijk\ell}\frac{\delta P^{1/2}}{\delta h_{k\ell}}\right) \tag{4.1}$$

This is related to an intimate connection between Q and Fisher information based on techniques of Hall and Reginatto (cf. [61, 62, 96]. The WDW equation is

$$\left[-\frac{\hbar^2}{2}\frac{\delta}{\delta h_{ij}}G_{ijk\ell}\frac{\delta}{\delta h_{k\ell}} + V\right]\psi = 0; \tag{4.2}$$

and there is a lovely relation

$$\int \mathfrak{D}hPQ = -\int \mathfrak{D}h\frac{\delta P^{1/2}}{\delta h_{ij}}G_{ijk\ell}\frac{\delta P^{1/2}}{\delta h_{k\ell}} \tag{4.3}$$

where the last term is Fisher information (cf. [28, 49, 61, 62, 96]).

(2) In [115] one uses again the attractive sandwich ordering of (4.2) (which is mandatory in (4.2)) and considers WDW in the form

$$\left[h^{-q}\frac{\delta}{\delta h_{ij}}h^q G_{ijk\ell}\frac{\delta}{\delta h_{k\ell}} + \sqrt{h}^{(3)}\mathcal{R} + \right. \tag{4.4}$$

$$\left. + \frac{1}{2\sqrt{h}}\frac{\delta^2}{\delta\phi^2} - \frac{1}{2}\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}\sqrt{h}V(\phi)\right]\psi = 0$$

with momentum constraint (**4C**) $i[2\nabla_j(\delta/\delta h_{ij}) - h^{ij}\partial_j\phi(\delta/\delta\phi)]\psi = 0$ where ϕ is a matter field, q is an ordering parameter, and $h = det(h_{ij})$. Putting this in "polar" form $\psi = \sqrt{\rho}exp(iS/\hbar)$ leads to

$$G_{ijk\ell}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{k\ell}} + \frac{1}{2\sqrt{h}}\left(\frac{\delta S}{\delta\phi}\right)^2 - \sqrt{h}({}^{(3)}\mathcal{R} - \mathcal{Q}_G) + \tag{4.5}$$

$$+ \frac{\sqrt{h}}{2}h^{ij}\partial_i\phi\partial_j\phi + \frac{\sqrt{h}}{2}(V(\phi) - \mathcal{Q}_M) = 0$$

where the gravity and matter quantum potentials are given via

$$\mathcal{Q}_G = -\frac{1}{\sqrt{\rho h}}\left(G_{ijk\ell}\frac{\delta^2\sqrt{\rho}}{\delta h_{ij}\delta h_{k\ell}} + h^{-q}\frac{\delta h^q G_{ijk\ell}}{\delta h_{ij}}\frac{\delta\sqrt{\rho}}{\delta h_{k\ell}}\right);\ \mathcal{Q}_M = -\frac{1}{h\sqrt{\rho}}\frac{\delta^2\sqrt{\rho}}{\delta\phi^2} \tag{4.6}$$

There is a continuity equation

$$\frac{\delta}{\delta h_{ij}}\left[2h^q G_{ijk\ell}\frac{\delta S}{\delta h_{k\ell}}\rho\right] + \frac{\delta}{\delta\phi}\left[\frac{h^q}{\sqrt{h}}\frac{\delta S}{\delta\phi}\rho\right] = 0 \tag{4.7}$$

and the momentum constraint leads to equations (**4D**) $2\nabla_j(\delta\sqrt{\rho}/\delta h_{ij})-h^{ij}\partial_j\phi(\delta\sqrt{\rho}/\delta\phi)=0$ and $2\nabla_j(\delta S/\delta h_{ij})-h^{ij}\partial_j\phi(\delta S/\delta\rho)=0$ while the Bohmian "guidance" equations are

$$\frac{\delta S}{\delta h_{ij}}=\pi^{k\ell}=\sqrt{h}(K^{k\ell}-h^{k\ell}K);\ \frac{\delta S}{\delta\phi}=\pi_\phi=\frac{\sqrt{h}}{N^\perp}\dot{\phi}-\sqrt{h}\frac{N^i}{N^\perp}\partial_i\phi \tag{4.8}$$

where K^{ij} is the extrinsic curvature. Since in the WDW equation the wavefunction is in the ground state with zero energy the stability condition of the metric and matter field is (**4E**) $h^{ij}\partial_i\phi\partial_j\phi+V(\phi)-2^3\mathcal{R}+\mathcal{Q}_M+2\mathcal{Q}_G=0$ which is a pure quantum solution (this follows from (4.5) by setting all functional derivatives of S to be zero). In [115] these equations are examined perturbatively and we refer to [105, 111] for discussion of the constraint algebra and related matters.

(3) In [111] one studies the constraint algebra and equations of motion based on a Lagrangian (**4F**) $\mathfrak{L}=\sqrt{-g}\mathcal{R}=\sqrt{h}N({}^{(3)}\mathcal{R}+Tr(K^2)-(TrK)^2)$ where ${}^{(3)}$is the 3-D Ricci scalar, K_{ij} the extrinsic curvature, and h the induced spatial metric. The canonical momentum of the 3-metric is given via (**4G**) $P^{IJ}=\partial\mathfrak{L}/\partial\dot{h}_{ij})=\sqrt{h}(K^{ij}-h^{ij}TrK)$ and the classical Hamiltonian is (**4H**) $H=\int d^3x\mathfrak{H}$ with $mfH=\sqrt{h}(NC+N^iC_i)$. Here one has

$$C=-{}^{(3)}\mathcal{R}+\frac{1}{h}\left(Tr(p^2)-\frac{1}{2}(Trp)^2\right)=-2G_{\mu\nu}n^{mu}n^\nu; \tag{4.9}$$

$$C_i=-2^{(3)}\nabla^j\left(\frac{p_{ij}}{\sqrt{h}}\right)=-2G_{\mu i}n^\mu$$

where n^μ is normal given via $n^\mu=(1/N,-\mathbf{N}/N)$. To get the quantum version one takes $H\to H+Q$ ($\mathfrak{H}\to\mathfrak{H}+\mathcal{Q}$ (where $Q=\int d^3x\mathcal{Q}$) and

$$\mathcal{Q}=\hbar^2NhG_{ijk\ell}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta h_{ij}\delta h_{k\ell}} \tag{4.10}$$

The classical constraints are then modified via $C\to C+(\mathcal{Q}/\sqrt{h}N)$ and $C_i\to C_i$. We disregard the constraint algebra here and go some formulas for quantum Einstein equations. First there is an HJ equation

$$G_{ijk\ell}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{k\ell}}-\sqrt{h}({}^{(3)}\mathcal{R}-\mathcal{Q})=0 \tag{4.11}$$

where S is the phase of the wave function and this leads to Bohm-Einstein equations

$$\mathcal{G}^{ij} = -\kappa\mathcal{T}^{ij} - \frac{1}{N}\frac{\delta(\mathcal{Q}_G + \mathcal{Q}_m}{\delta g_{ij}};\ \mathcal{G}^{0\mu} = -\kappa\mathcal{T}^{0\mu} + \frac{\mathcal{Q}_G + \mathcal{Q}_m}{2\sqrt{-g}}g^{0\mu}; \tag{4.12}$$

$$\mathcal{Q}_m = \hbar^2\frac{N\sqrt{H}}{2}\frac{\delta^2|psi|}{\delta\phi^2};\ \mathcal{Q}_G = \hbar^2 NhG_{ijk\ell}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta h_{ij}\delta h_{k\ell}}$$

These are the quantum version of the Einstein equations and since regularization here only affects the quantum potential (cf. [111]) for any regularization the quantum Einstein equations are the same and one can write (**4I**) $\mathcal{G}^{\mu\nu} = -\kappa\mathcal{T}^{\mu\nu} + \mathfrak{S}^{mu\nu}$ with

$$\mathfrak{S}^{0\mu} = -\frac{\mathcal{Q}_G + \mathcal{Q}_m}{2\sqrt{-g}}g^{0\mu} = \frac{\mathcal{Q}}{2\sqrt{-g}}g^{0\mu};\ \mathfrak{S}^{ij} = -\frac{1}{N}\frac{\delta Q}{\delta g_{ij}} \tag{4.13}$$

(4) There are also developments of Bohmian theory and quantum geometrodynamics in [5, 6, 10, 37, 74, 92, 93, 104, 117] (cf. [21] for some survey and more references). In [92] for example one writes the WDW equation in the form

$$\left\{-\hbar^2\left[\kappa G_{ijk\ell}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{k\ell}} + \frac{1}{2}h^{-1/2}\frac{\delta^2}{\delta\phi^2}\right] + \right\}\psi(h_{ij},\phi) = 0; \tag{4.14}$$

$$V = h^{1/2}\left[-\kappa^{-1}(\mathcal{R}^{(3)} - 2\Lambda) + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + U(\phi)\right]$$

(questions of factor ordering and regularization are ignored here) with a constraint (**4J**) $-2h_{ij}\nabla_j(\delta\psi/\delta h_{ij}) + (\delta\psi)/\delta\phi)\partial_i\phi = 0$. Writing now $\psi = Rexp(iS/\hbar)$ (**4J**) leads to

$$-2h_{ij}\nabla_j(\delta S/\delta h_{ij}) + (\delta S/\delta\phi)\partial_i\phi = 0;\ -2h_{ij}\nabla_j(\delta R/\delta h_{ij}) + (\delta R/\delta\phi)\partial_i\phi = 0 \tag{4.15}$$

and (4.14) yields

$$\kappa G_{ijk\ell}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{k\ell}} + \frac{1}{2}h^{-1/2}\left(\frac{\delta S}{\delta\phi}\right)^2 + V + Q = 0; \tag{4.16}$$

$$Q = -\frac{\hbar^2}{R}\left(\kappa G_{ijk\ell}\frac{\delta^2 R}{\delta h_{ij}h_{j\ell}} + \frac{h^{-1/2}}{2}\frac{\delta^2 R}{\delta\phi^2}\right);$$

$$\kappa G_{ijk\ell}\frac{\delta}{\delta h_{ij}}\left(R^2\frac{\delta S}{\delta h_{k\ell}}\right) + \frac{1}{2}\frac{\delta}{\delta\phi}\left(R^2\frac{\delta S}{\delta\phi}\right) = 0$$

(5) In [107] one picks up again the approach of (2) to find a pure quantum state leading to a static Einstein universe whose classical counterpart is flat spacetime. For WDW one uses a form of (4.4) (with $16\pi G = 1$ and $\mathcal{R}$ the 3-curvature scalar), namely

$$\hbar^2 h^{-q}\frac{\delta}{\delta h_{ij}}\left(h^q G_{ijk\ell}\frac{\delta\psi}{\delta h_{k\ell}}\right)+\sqrt{h}\mathcal{R}\psi+\frac{1}{\sqrt{h}}\mathcal{T}^{00}\left(\frac{-i\hbar\delta}{\delta\phi_a},\phi_a\right)\psi=0; \tag{4.17}$$

with 3-diffeomorphism constraint in the form (**4K**) $2\nabla_j(\delta/\delta h_{ij})\psi - \mathcal{T}^{i0}(\delta/\delta\phi_a,\phi_a)\pi = 0$. $\mathcal{T}^{\mu\nu}$ is the energy momentum tensor of matter fields ϕ_a in which the matter is quantized by replacing its conjugate momenta by $-i\hbar\delta/\delta\phi_a$. For the causal interpretation one sets again $\psi = Rexp(iS/\hbar)$ to obtain

$$G_{ijk\ell}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{k\ell}}-\sqrt{h}(\mathcal{R}-Q_G)+\frac{1}{\sqrt{h}}(\mathcal{T}^{00}(\delta S/\delta\phi_a,\phi_a)+Q_M)=0; \tag{4.18}$$

$$\frac{\delta}{\delta h_{ij}}\left(2h^q G_{ijk\ell}\frac{\delta S}{\delta h_{k\ell}}R^2\right)+\sum\frac{\delta}{\delta\phi_a}\left(h^{q-(1/2)}\frac{\delta S}{\delta\phi_a}R^2\right)=0 \tag{4.19}$$

$$Q_G=-\frac{\hbar^2}{\sqrt{h}R}\left(h^{-q}\frac{\delta}{\delta h_{ij}}h^q G_{ijk\ell}\frac{\delta R}{\delta h_{k\ell}}\right);\ Q_M-\frac{\hbar^2}{hR}\sum\frac{\delta^2 R}{\delta\phi_a^2} \tag{4.20}$$

$$2\nabla_j\frac{\delta R}{\delta h_{ij}}-\mathcal{T}^{i0}(\delta R/\delta\phi_a,\phi_a)=0;\ 2\nabla_j\frac{\delta S}{\delta h_{ij}}-\mathcal{T}^{i0}(\delta S/\delta\phi_a,\phi_a)=0 \tag{4.21}$$

One notes that all terms containing the second functional derivative are ill defined and can be regulated via $(\delta/\delta h_{ij}(x))(\delta/\delta h_{ij}(x) \to \int d^3x\sqrt{h}U(x-x')(\delta/\delta h_{ij}(x))(\delta/\delta h_{ij}(x'))$ where U is the regulator. Finally the guidance equations are (**4L**) $\pi^{k\ell} = \sqrt{h}(K^{k\ell} - Kh^{k\ell}) = \delta S/\delta h_{k\ell}$ and $\pi_{\phi_a} = \delta S/\delta\phi_a$ where $K_{ij} = (1/2N)(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i)$ is the extrinsic curvature. Using the quantum Hamilton-Jacobi-Einstein (HJE) equation one can define a limit, called the pure quantum limit, where the total quantum potential is of the same order as the total classical potential and they can cancel each other. In this case one has (**4M**) $\delta S/\delta h_{ij} = \delta S/\delta\phi_a = 0$ and the continuity is satisfied identically. The resulting trajectory is not similar to any classical solution and the quantum HJE equation for a pure quantum state is an equation for spatial dependence of the metric and matter fields in terms of the norm of the wave function. Explicit calculations are given for some special situations.

REMARK 4.1. Note that for $\eta_{ab} \sim (1,-1,-1,-1)$ and $\hbar = c = 1$ one has $\partial_0^2 - \nabla^2 \sim \Box$ and (**4N**) $(\nabla S)^2 = m^2[1 + (\Box R/m^2R)]$ as in (3.4) (Nikoli'c).

This agrees with

$$(4.22)\qquad (\nabla S)^2 = \mathfrak{M}^2c^2(1+Q);\ Q = \frac{\hbar^2}{m^2c^2}\frac{\Box R}{R}$$

from Section 3.1 (F. and A. Shojai). For the BFM theory with $\eta_{ab} \sim (-1,1,1,1)$ one has (**4O**) $(1/2m)(\nabla S)^2 + (mc^2/2) - (\hbar^2/2m)(\Box R/R) = 0$ from (3.2). But $\Box R \to -\Box R$ and $(\nabla S)^2 \sim \eta^{ab}\nabla_b S\nabla_a S \to -(\nabla S)^2$ for $\eta_{ab} \to -\eta_{ab}$. Hence in the $\eta_{ab} = (1,-1,-1,-1)$ notation one obtains $(\nabla S)^2 = m^2c^2[1 + (\hbar^2/m^2c^2)(\Box R/R)]$ as in (4.22). ∎

REMARK 4.2. There is a lot of motivation here for using the quantum potential as a generator of quantum gravity (cf. [20]) and also for considering the conformal factor $\mathfrak{M}^2/m^2$ as a generator of Ricci flow (cf. [39, 56, 91]). ∎

REMARK 4.3. One finds fascinating connections between Bohmian theory and phase space mechanics in [14, 15, 53, 54, 55, 79, 80, 81, 119]. In [14] one argues that if the quantum potential (QP) reflects the quantum aspects of a sysem it should be possible to to identify such aspects within the QP and in particular one shows how the balance between localisation and dispersion energies suggests a link between the QP and the Heisenberg uncertainty principle. Recall first that from the SE (**5A**) $i\hbar\partial_t\psi = [-(\hbar^2/2m)\nabla^2 + V]\psi$ there follows

$$(4.23)\qquad \partial_t S + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{r} + V = 0;\ \partial_t\rho + \nabla\cdot\left(\rho\frac{p}{m}\right) = 0$$

where $\psi = Rexp(iS/\hbar)$, $\rho = |\psi|^2$, and $p = \nabla S$. The QP is manifestly of the form $Q = -(\hbar^2/2m)(\nabla^2 R/R)$ and one writes $F = -\nabla(Q+V)$ and $v = j/\rho = p/m$. Now consider a more general derivation of the QP by writing the SE in the form (**5B**) $i\hbar\partial_t\psi = (T(\hat{p}) + V(\hat{x}))\psi$. Setting again $\psi = Rexp(iS/\hbar)$ one obtains

$$(4.24)\qquad \partial_t S + \Re\left(\frac{T\psi}{\psi}\right) + V(x) = 0;\ \partial_t\rho - \frac{2\rho}{\hbar}\Im\left(\frac{T\psi}{\psi}\right) = 0$$

Correspondingly in the momentum space with $\hat{x} = i\hbar\nabla_p$ and $\hat{p} = p$ the real and imaginary parts of the SE are

$$(4.25)\qquad \partial_t S + T(p) + \Re\left(\frac{V\psi}{\psi}\right) = 0;\ \partial_t\rho - \frac{2\rho}{\hbar}\Im\left(\frac{V\psi}{\psi}\right) = 0$$

Then expanding exponentials one writes

$$(4.26)\qquad \Re\left(\frac{\psi^* T\psi}{\rho}\right) = \Re\left(\frac{R[1-(iS/\hbar)-\cdots)T(\hat{p})R(1+(iS/\hbar)-\cdots]}{\rho}\right)$$

(note the formal equivalence $T\psi/\psi = \psi^* T\psi/\rho$). If now $T(\hat{p})$ is a general but analytic function of $\hat{p}$ one can expand in a power series in $\hat{p} = -i\hbar\nabla$

and the kinetic term may be separated into the sum of two parts

$$\Re\left(\frac{T\psi}{\psi}\right) = T_h(x) + T_0(x);\ T_0(x) = T(\nabla S) \tag{4.27}$$

where $T_h(x)$ is an expansion in even positive powers of $\hbar$ and $T_0(x)$ is independent of $\hbar$ and identifies $p = \nabla S$. The same line of argument allows the potential term of the HJ equation in (4.25) to be separated as

$$\Re\left(\frac{V\psi}{\psi}\right) = V_h(p) + V_0(p);\ V_0(p) = V(-\nabla_p S) \tag{4.28}$$

where V_h is an expansion in even positive powers of $\hbar$ and $V_0(p)$ is independent of $\hbar$ and identifies $x = -\nabla_p S$. We pursue this further in [30]. ■

5. THE QUANTUM POTENTIAL AND GEOMETRY

We begin with [105] and recall some features of Weyl geometry (some of which are indicated already in previous sections). We remember first that vectors change in length and direction under translation via (**5A**) $\delta\ell = \phi_\mu \delta x^\mu \ell$ so $\ell = \ell_0 exp(\int \phi_\mu dx^\mu)$ where ϕ_μ is the Weyl vector. Equivalently (**5B**) $g_{\mu\nu} \to exp(2\int \phi_\mu \delta x^\mu) g_{\mu\nu}$ which is a conformal transformation. Recall also that the metric is a Weyl covariant object of weight 2 and the Weyl connection is given via

$$\Gamma^\mu_{\nu\lambda} = \left\{ \begin{array}{c} \mu \\ \nu\lambda \end{array} \right\} + g_{\nu\lambda}\phi^\mu - \delta^\mu_\nu \phi_\lambda - \delta^\mu_\lambda \phi_\nu \tag{5.1}$$

A gauge transformation (**5C**) $\phi_\mu \to \phi'_\mu = \phi_\mu + \partial_\mu \Lambda$ transforms $g_{\mu\nu} \to g'_{\mu\nu} = exp(2\Lambda) g_{\mu\nu}$ with $\delta\ell \to \delta\ell' = \delta\ell + (\partial_\mu \Lambda) dx^\mu \ell$. In remarks just before Section 3.1 we have seen how quantum effects are geometrized via the Dirac field β and gauge transformations. Let us now make more explicit some direct relations between the quantum potential and geometric ideas via the Weyl vector. We recall from Section 3 (#2) that for the SE $Q = -(m/2)\mathbf{u}^2 - (\hbar/2)\partial\mathbf{u}$ where $\mathbf{u}$ is an osmotic velocity (see also [21, 50]). Similarly for the KG equation one has $Q = (\hbar^2/m^2c^2)(\Box_g|\psi|/|\psi|$ for example as in Remark 4.1 and we will also consider an appropriate osmotic velocity for this situation.

Consider now the situation $Q = 0$ for the SE which can be expressed in several forms.

(1) Defining the osmotic velocity as $\mathbf{u} = D\nabla log(\rho)$ with $D = \hbar/2m$ from [21, 50] (cf. also [86, 87]) one has then (**5A**) $(m/2)\mathbf{u}^2 + (\hbar/2)\nabla\mathbf{u} = 0$.
(2) Another form is directly (for $g = 1$) (**5B**) $\Delta\sqrt{\rho} = 0$.

(3) There is a general form for (**5C**) $\phi_i = -\partial_i log(\hat{\rho})$ with $\hat{\rho} = \rho/\sqrt{g}$, namely

$$\dot{\mathcal{R}} + 2\left[\phi_i\phi^i - \frac{2}{\sqrt{g}}\partial_i(\sqrt{g}\phi^i)\right] = 0 \tag{5.2}$$

When $\dot{\mathcal{R}} = 0$ with $\sqrt{g} = 1$ and $\hat{\rho} = \rho$ this becomes (**5D**) $\phi_i\phi^i - 2\partial_i\phi^i = 0$. Note $\phi^i \sim g^{ik}\phi_k = -g^{ik}\partial_k log(\hat{\rho}) = -\partial^k log(\hat{\rho})$.

(4) From (2.8) another form of (5.2) above is

$$\dot{\mathcal{R}} + \frac{8}{\sqrt{\rho}}\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\hat{\rho}}) = 0 \tag{5.3}$$

(5) In view of [21, 25, 39, 61, 62, 96] one can say that the fundamental quantum fluctuation or perturbation in momentum has the form (**5E**) $\delta p \sim c(\nabla\rho/\rho)$ and this means (**5F**) $\delta p \sim \hat{c}\mathbf{u}$ or equivalently $\delta p \sim \tilde{c}\vec{\phi}$. We can assume that in a Weyl space situation an osmotic velocity $\mathbf{u} = Dlog(\hat{\rho})$ is meaningful. The "obligatory" nature of $\delta p \sim c(\nabla\rho/\rho)$ is made even more striking in the developments in [65, 78]. One shows there in particular that a classical momentum can be written as

$$\hat{p}_{cl} = \hat{p} + \left(\frac{i\hbar}{2}\right)\left(\frac{\nabla\rho}{\rho}\right) \Rightarrow \hat{p}_{cl} = -i\hbar\left(\nabla - \frac{1}{2}\frac{\nabla(\psi^*\psi)}{\psi^*\psi}\right) \tag{5.4}$$

It should now be possible to extract some analytic and geometric features of the situation $Q = 0$.

EXAMPLE 5.1. We think of $\psi = \sqrt{\rho}exp(iS/\hbar)$ with $\sqrt{\rho} = R$. Take #2 first and look for solutions of $\Delta R = 0$ in a finite region Ω with $R \in H_0^1(\Omega)$ (Sobolev space) for example (see [31, 44] for techniques and results in PDE). For a QM situation $R = 0$ on $\partial\Omega$ and $H_0^1(\Omega)$ is the natural setting with $R \in L^2(\Omega)$. However by Green's theorem $\int_\Omega R\Delta R dV = -\int_\Omega |\nabla R|^2 dS = 0$ which implies $\nabla R = R = 0$. this is consistent with the Example 1.2 where ψ involves plane waves and L^2 solutions are meaningless. ■

EXAMPLE 5.2. Consider next a situation $(m/2)|\mathbf{u}|^2 + (\hbar/2)\nabla\mathbf{u} = 0$ or in 1-D $\partial u + cu^2 = 0$ with $c > 0$. Then $u'/u^2 = -c \Rightarrow u = (\hat{c} + cx)^{-1}$ and setting $u = D\rho'/\rho$ yields $R^2 = \rho = k(\hat{c} + cx)^d$. This is not reasonable for $R = 0$ outside of a finite Ω. ■

EXAMPLE 5.3. Consider $\phi_i\phi^i - 2\partial\phi^i = 0$ or equivalently (in 1-D for convenience) $\phi'/\phi^2 = 1/2$ leading to $-(1/\phi) = (1/2)x + c$ and problems similar to those in Example 5.2. ■

We can however think of ρ, $\vec{\phi}$, or $\mathbf{u}$ as functions of Q so for each admissible Q there will be in principle some well determined R, modulo spectral conditions as in Remark 1.1.

REMARK 5.1. In Remark 1.1 we saw that determining R from Q involved solving $\Delta R + \beta QR = 0$ ($\beta > 0$) in say $H^1_0(\Omega)$. If $Q \leq 0$ this yields a unique solution while if 0 is not in the spectrum of $\Delta + \beta Q$ then (**5G**) $\Delta R + \beta QR = 0$ has a unique solution for say $Q \in L^\infty(\Omega)$. We also saw that modulo solvability of (1.4) one would obtain a "generalized" quantum theory based on Q. We can improve the statement of this in Remark 1.1 by saying that, given solutions V and S of (1.4) (via a solution R of (**5G**)), in converting this to a SE one eliminates Q from the picture entirely. For R unique $S(x,t)$ is determined up to a function $f(t)$ and a function $g(x)$ arising from

$$S = -\int^t (Q+V)dt - \frac{1}{2R^4}\left[f(t) - \int^x \partial_t R^2 dx\right]^2 + g(x) \tag{5.5}$$

However V_x is known via (1.4) in terms of S_{xt} which depends only on Q (via R), f, and f', hence only in terms of one function $f(t)$. If then $V = V(x)$ it may actually be almost determined and Q, instead of determining only one trajectory based on R and S, actually could lead to the SE itself (modulo f) for $\psi = Rexp(iS/\hbar)$; if it were to be the case that $V = V(x)$ does not use $f(t)$ this means that Q alone would determine a "generalized" quantum theory via the SE! This could eliminate some of the ambiguity connected with the idea of using Q as a quantization. It would be worthwhile checking the equations to find such situations (see below). We note also that if Q contains t it is transmitted to R as a parameter in solving the elliptic equation; if Q is independent of t then of course so is R and this could conceivably simplify matters in determining $V = V(x)$. ■

We check this last idea in more detail now. Assume Q is a function of x alone, $Q = Q(x)$, and let it determine a unique $R \in H^1_0(\Omega)$ (normalized so that $\int_\Omega R^2 dx = 1$). Then look at (1.4), namely

$$S_t + \frac{1}{2m}S_x^2 + Q + V = 0;\ \partial_t R^2 + \frac{1}{m}(R^2 S_x)_x = 0 \tag{5.6}$$

The second equation becomes $(R^2S_x)_x = 0$ which implies (**5H**) $R^2 S_x = f(t)$ for some "arbitrary" $f(t)$. Then (**5I**) $S_t + (1/2m)(f^2/R^4) + Q + V = 0$ and we can eliminate S from (**5H**) and (**5I**) via

$$R^2 S_{xt} = f_t;\ S_{xt} - \frac{2f^2 R_x}{mR^5} + Q_x + V_x = 0 \Rightarrow \frac{2f^2}{m}\frac{R_x}{R^3} - f_t = R^2(Q_x + V_x) \tag{5.7}$$

This determines V_x in terms of $Q(x)$ and $f(t)$ so we ask whether $V = V(x)$ can occur (no t dependence). In such a case the t derivatives of the last term in (5.7) are zero yielding (**5J**) $f_{tt} = (2R_x/mR^3)\partial_t f^2$. This means

$$\frac{f_{tt}}{\partial_t f^2} = F(t) = \frac{2R_x}{mR} = \mathfrak{F}(x) \tag{5.8}$$

Consequently $F(t) = \mathfrak{F}(x) = c$ and (**5K**) $f_{tt} = c\partial_t f^2$ while $(R_x/R^3) = (cm/2)$ leading to

$$f_t = cf^2 + \hat{c};\ R^2 = \frac{1}{\tilde{c} - cmx} \tag{5.9}$$

Thus $x > (\tilde{c}/cm)$ but $R \notin H_0^1(\Omega)$ for any Ω. This seem to preclude $V = V(x)$ (or perhaps $Q = Q(x)$). Hence from (5.6) one has at least (cf. also Section 9)

PROPOSITION 5.1. Given $Q = Q(x)$ determining a unique $R(x) \in H_0^1(\Omega)$ it follows that V_x is determined up to an "arbitrary" function $f(t)$ via

$$V_x = \frac{1}{R^2}\left[\frac{2f^2R_x}{mR^3} - f_t\right] - Q_x \tag{5.10}$$

This situation precludes V being a function of x alone. ■

REMARK 5.2. Note if $\int_\Omega R^2(x,t)dx = r^2(t)$ then to get a proper normalization one would take $\mathcal{R}(x,t) = (1/r(t))R(x,t)$ and note that Q computed on $\mathcal{R}$ is equal via (**5L**) $\mathcal{Q} = -(\hbar^2/2m)(\mathcal{R}_{xx}/\mathcal{R}) = -(\hbar^2/2m)(R_{xx}/R)$. Note also that r is determined by Q via R. We still think of $\psi \sim Rexp(iS/\hbar)$ so (5.5) applies and (**5H**) becomes (**5M**) $RS_x^2 = -m\int^x \partial_t R^2 dx + f(t) = A(f,Q)$ (since Q determines R). Then we are still essentially in the context of Remark 5.1 and (**5N**) $2RR_tS_x + R^2S_{xt} = \partial_t A$ while from (5.5) (**5O**) $S_{xt} + (1/m)S_xS_{xx} + Q_x + V_x = 0$. Now we eliminate S_{xx} and S_{xt} to get V_x in terms of Q and f. First from (**5N**) one has

$$S_{xt} = \frac{1}{R^2}\left\{\partial_t A - 2\frac{R_t}{R}A\right\} \tag{5.11}$$

while (**5P**) $R^2S_{xx} + 2RR_x = \partial_x A \Rightarrow S_{xx} = (1/R^2)(\partial_x A - 2RR_x)$. Hence one arrives at

$$\frac{1}{R^2}\left[\partial_t A - \frac{2AR_t}{R}\right] = -Q_x - V_x - \frac{1}{m}\frac{A}{R^4}(\partial_x A - 2RR_x) \tag{5.12}$$

and we can state

PROPOSITION 5.2. Defining $A(f,Q) = f(t) = m\int^x \partial_t R^2 dx$ with f "arbitrary" one can determine V_x via (5.12) as $V_x(Q,f)$. Hence $V = \int^x V_x dx + h(t)$ for h "arbitrary" provides a potential $V(Q,f,h)$ and the associated SE is determined completely by V. If choices $f = h = 0$ are "natural" one can say that Q determines a natural SE and a corresponding "generalized" quantum theory. ■

REMARK 5.3. Consider the stationary case (cf. [21]) (**5Q**) $(1/2m)S_x^2 + Q + V - E = 0$ with $(R^2S_x)_x = 0$ where $R = R(x)$ is say uniquely determined via $Q = Q(x)$ (note however that both R and Q must contain E as

a parameter). Then

$$S_x = \frac{c}{R^2};\ \frac{1}{2m}\left(\frac{c^2}{R^4}\right) + Q + V - E = 0 \tag{5.13}$$

This means (**5R**) $1 = \partial_E Q - (c^2/mR^5)\partial_E R$ (since V does not depend on E) and hence

$$\frac{2c^2 R_E}{mR^5} + 1 = \frac{\hbar^2 R'' R_E}{2mR^2} - \frac{\hbar^2 R''_E}{2mR} \tag{5.14}$$

Viewed in terms of $\rho = R^2$ this mean that $\rho = \rho(E, x)$ and the corresponding Weyl geometry based on $\vec{\phi} = -\nabla log(\rho)$ will depend on E (as will Q of course). We refer here also to the quantum mass idea of Floyd, namely $m_Q = m(1 - \partial_E Q)$ for stationary situations (this is sketched in [32] for example and we refer to [48] for more details). Some further ideas about this are sketched in Remark 7.3. In particular one knows that $Q \sim -(\hbar^2/2m)\mathcal{R}$ (from Section 2) where $\mathcal{R}$ is the Ricci-Weyl curvature with $\hbar$ essentially put in by hand to conform to the wave function idea and operator QM. We see that the geometry of the space in which a trajectory transpires is thereby determined by E (not surprisingly) which seems to say that the probability distribution ρ is the basic unknown here (and in the time dependent situation). Once one has a probability distribution one can posit a wave function and insert $\hbar$. In fact (given V) the two equations $(1/2m)(c^2/R^2) + Q + V - E = 0$ and $Q = -(\hbar^2/2m)(R''/R)$ determine $R = R(E, x)$ directly ($\hbar$ being gratuitously inserted). ∎

6. OLAVO THEORY

We go here to [89] and sketch some matters dealing with uncertainty and the SE (cf. also [12, 32]). In the first paper of [89] an axiomatic formulation for quantum mechanics (QM) is given. Consider ensembles described by probability density functions in phase space described as $F(x, p, t)$; assume

(1) Newtonian particle mechanics is valid for particles in the ensemble.
(2) For an isolated system $dF(x, p, t)/dt = 0$.
(3) The Wigner-Moyal infinitesimal transformation is defined via

$$Z_Q(x, \delta x/2, t) = \int F(x, p, t) exp\left(\frac{ip\delta x}{\ell}\right) dp \tag{6.1}$$

where ℓ is a parameter which will necessarily be equal to $\hbar$.

From these axioms one can derive nonrelativistic QM as follows. First using (1) and (2) one has

$$\frac{dF}{dt} = \partial_t F + \dot{x} F_x + \dot{p} F_p = 0;\ \dot{x} = \frac{p}{m};\ \dot{p} = f = -V_x \tag{6.2}$$

Multiplying by the exponential in (6.1) and integrating one arrives at

$$-\partial_t Z_Q + \frac{i\hbar}{m}\frac{\partial^2 Z_Q}{\partial x\partial(\delta x)} - \frac{i}{\hbar}\delta V(x) Z_Q = 0 \tag{6.3}$$

where the infinitesimal nature of δx is used to write

$$\partial_x V\delta x = \delta V(x) = V\left(x + \frac{\delta x}{2}\right) - V\left(x - \frac{\delta x}{2}\right); \tag{6.4}$$

and one knows that ($\bigstar$) $F(x,p,t)exp\left(\frac{ip\delta x}{\ell}\right)_{p=-\infty}^{p=\infty} = 0$ by the nature of probability distributions. Writing $y = x + (\delta x/2)$ and $y' = x - (\delta x/2)$ one can rewrite (6.3) as

$$\left\{\frac{\hbar^2}{2m}[\partial_y^2 - \partial_{y'}^2] - [V(y) - V(y')]\right\} Z_Q(y,y',t) = -i\hbar\partial_t Z_Q(yy',t) \tag{6.5}$$

This is called a Schrödinger equation (SE) for the characteristic function Z_Q and is valid for all values of y, y' as long as they are infinitesimally close.

Now suppose one can write (this is a fundamental assumption)

$$Z_Q(y,y',t) = \psi^*(y',t)\psi(y,t);\ \ \psi(y,t) = R(t,y)e^{iS(y,t)/\hbar} \tag{6.6}$$

Then expanding Z_Q one obtains
(6.7)

$$Z_Q(y,y',t) = \left\{R(x,t)^2 - \left(\frac{\delta x}{2}\right)^2\left[(\partial_x R)^2 - R\frac{\partial^2 R}{\partial x^2}\right]\right\} exp\left(\frac{i}{\hbar}(\delta x)S_x\right)$$

Putting this in (6.3) leads to

$$P_t + \partial_x\left(\frac{PS_x}{m}\right) = 0;\ \ \frac{i\delta x}{\hbar}\partial_x\left[\frac{S_x^2}{2m} + V + S_t - \frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial x^2}\right] = 0 \tag{6.8}$$

where $P(x,t) = R^2 = lim_{\delta x\to 0} Z_Q(x+(\delta x/2), x-(\delta x/2), t)$ is the probability distribution in configuration space. Equation (6.8b) can be rewritten as

$$\frac{1}{2m}S_x^2 + V + S_t + Q = f(t);\ \ Q = -\frac{\hbar^2}{2mR}\partial_x^2 R \tag{6.9}$$

One can eliminate $f(t)$ by redefining $S(x,t)$ as $S'(x,t) = S(x,t) + \int_0^t f(t')dt'$; since this is only a new definition of the energy reference level one can simply take $f(t) = 0$ and obtain as well (**6A**) $(\hbar^2/2m)\psi_{xx} - V(x)\psi = -i\hbar\psi_t$. Thus if one can write Z_Q as the product (6.6) then ψ satisfies the SE (**6A**) and (6.8) holds.

Now define operators on Z_Q via primes (distinguished from operators

acting on the probability amplitude) so that $\hat{p}' = -i\hbar(\partial/\partial(\delta x))$ and $\hat{x}' = x$; this is based on the fact that

$$\bar{p} = \lim_{\delta x \to 0} -i\hbar \frac{\partial}{\partial(\delta x)} \int F(x,p,t) exp\left(i\frac{p\delta x}{\hbar}\right) dxdp \tag{6.10}$$

and $\bar{x} = lim_{\delta x \to 0} \int xF(x,p,t)exp[ip\delta x/\hbar]dxdp$. Thus the result of position and momentum operators acting on Z_Q represents a mean value calculation for the ensemble components. Observe now via (6.7) that

(6.11)

$$\bar{p} = lim_{\delta x \to 0} \left[-i\hbar \frac{\partial}{\partial(\delta x)} \int F(x,p,t) exp\left(i\frac{p\delta x}{\hbar}\right) dxdp\right] = \int R^2(x) S_x dx$$

One can rewrite (6.11) in the form

$$\hat{p} = lim_{\delta x \to 0} \left\{-i\hbar \int \frac{\partial}{\partial(\delta x)} \left[\psi^*\left(x - \frac{\delta x}{2}, t\right)\psi\left(x + \frac{\delta x}{2}, t\right)\right] dx\right\} \tag{6.12}$$

Again via (6.7) this leads to (**6B**) $\hat{p} = \int \psi^* x, t)(-i\hbar\partial_x)\psi(x,t)dx$ with similar calculations for the position operator. Consequently (**6C**) $\hat{p}\psi = -i\hbar\partial_x\psi$ and $\hat{x}\psi = x\psi$. Moreover some calculation shows that $[\hat{x},\hat{p}] = i\hbar$ and consequently $\Delta x \Delta p \geq \hbar/2$ (cf. [89] for details).

Next we go to the third paper in [89] to sketch another derivation of the SE from more physically grounded axioms and then the derivations are connected. Thus begin with the Liouville equation for $F(x,p,t)$

$$\partial_t F + \frac{p}{m}\partial_x F - \partial_x V \partial_p F = 0 \tag{6.13}$$

Integrating in p and using the definitions (**6D**) $\int F dp = \rho(x,t)$ and $\int pF dp = p(x,t)\rho(x,t)$ one obtains

$$\partial_t \rho + \partial_x \left[\frac{p(x,t)}{m}\rho(x,t)\right] = 0 \tag{6.14}$$

Then multiply the Liouville equation by p and integrate in order to obtain (**6E**) $\int p^2 F(x,p,t)dp = M_2(x,t)$ with

$$\partial_t[\rho(x,t)p(x,t)] + \frac{1}{m}\partial_x M_2 + (\partial_x V)\rho(x,t) = 0 \tag{6.15}$$

Putting (6.14) into (6.15) one gets after some calculation

(6.16)

$$\frac{1}{m}\partial_x[M_2(x,t) - p^2(x,t)\rho(x,t)] + \rho(x,t)\left[\partial_t p(x,t) + \partial_x\left(\frac{p^2(x,t)}{2m}\right) + V_x\right] = 0$$

One can write then

(6.17)

$$M_2 - p^2(x,t)\rho(x,t) = \int [p^2 - p^2(x,t)]F(x,p,t)dp = \int [p - p(x,t)]^2 F(x,p,t)dp$$

and set (**6F**) $\overline{(\delta p)^2}\rho(x,t) = \int [p - p(x,t)]^2 F(x,p,t)dp$. Then one wants to find a functional expression for $\overline{(\delta p)^2}$ and we note here that this expression is related to the momentum fluctuations used in [21, 61, 62, 96]. Thus consider the entropy of an isolated system in the form (**6G**) $\mathfrak{S}(x,t) = k_B log(\Omega(x,t))$ where k_B is the Boltzman constant and $\Omega(x,t)$ represents the system accessible states when the position x varies between x and $x + \delta x$. The equal a priori probability postulate then says that (**6H**) $\rho(x,t) \propto \Omega(x,t) = exp(\mathfrak{S}(x,t)/k_B)$ (cf. [98]). One can now consider entropy defined on configuration space and write (**6I**) $\mathfrak{S} = \mathfrak{S}_{eq} + (1/2)(\partial^2\mathfrak{S}_{eq}/\partial x^2)(\delta x)^2$ where $\mathfrak{S}_{eq}(x)$ is the statistical equilibrium configuration entropy; here one has used the fact that the entropy must be a maximum giving $(\partial\mathfrak{S}_{eq}/\partial x) = 0$ for $\delta x = 0$. One could also have divided the system into N cells of dimension δx and written (each i refers to one specific cell)

$$\mathfrak{S}_i = (\mathfrak{S}_{eq})_i + (\partial_x\mathfrak{S}_{eq})\delta x_i + \frac{1}{2}\left(\frac{\partial^2\mathfrak{S}_{eq}}{\partial x^2}\right)(\delta x_i)^2; \tag{6.18}$$

Using known properties of entropy one can write ($\sum_1^N \delta x_i = 0$ and the system is adiabatically isolated)

$$\mathfrak{S} = \sum_1^N \mathfrak{S}_i = \mathfrak{S}_{eq} + \frac{1}{2}\left(\frac{\partial^2\mathfrak{S}}{\partial x^2}\right)\sum_1^N(\delta x_i)^2 \tag{6.19}$$

Now use (**6I**) to get

$$\tilde{\rho}(x,\delta x,t) = \rho_{eq}(x,t)exp\left(-\frac{1}{2k_B}\left|\frac{\partial^2\mathfrak{S}_{eq}(x,t)}{\partial x^2}\right|(\delta x)^2\right) \tag{6.20}$$

(recall that the second derivative of the entropy is negative near an equilibrium point).

Thus at each point x the probability distribution with respect to small displacements $\delta x(x,t)$ is Gaussian and is related with the probability of having a fluctuation $\Delta\rho$ owing to a fluctuation δx. (6.20) guarantees that the system will tend to return to its equilibrium distribution represented by ρ_{eq} and using (6.20) the mean quadratic displacements related with the fluctuations are given via

$$\overline{(\delta x)^2} = \frac{\int_{-\infty}^{\infty}(\delta x)^2 exp(-\gamma(\delta x)^2)d(\delta x)}{\int_{-\infty}^{\infty} exp(-\gamma(\delta x)^2 d(\delta x)} = \frac{1}{2\gamma} \tag{6.21}$$

where (**6J**) $(1/2\gamma) = k_B|\partial^2\mathfrak{S}_{eq}(x,t)/\partial x^2|^{-1}$. Note here that x is a constant so this is correct; it is δx which is variable in the integration. A priori there is no relation between the displacement and momentum fluctuations but in the statistical equilibrium situation one can impose the restriction that (**6K**) $\overline{(\delta p)^2}\cdot\overline{(\delta x)^2} = \hbar^2/4$ (compare here with the exact uncertainty

principle of Hall and Reginatto in [21, 61, 62, 96]). Using (6.21) and (**6K**) gives then

(6.22)
$$\overline{(\delta p)^2} = -\frac{\hbar^2}{4}\frac{\partial^2 log(\rho(x,t))}{\partial x^2} \Rightarrow \overline{(\delta p)^2}\rho(x,t) = -\frac{\hbar^2}{4}\rho(x,t)\frac{\partial^2 log(\rho(x,t))}{\partial x^2}$$

Putting this in (6.16) and setting $\rho = R^2$ and $p = S_x$ one arrives at

$$R^2\partial_x\left[S_t + \frac{1}{2m}S_x^2 + V + Q\right] = 0 \tag{6.23}$$

which together with (6.14) is equivalent to the SE as before, namely

$$-\frac{\hbar^2}{2m}\psi_{xx} + V\psi = i\hbar\psi_t;\ \psi = Rexp(iS/\hbar) \tag{6.24}$$

Thus (6.14) and (6.16) have the same content as the SE and one notes that $p \sim S_x$ is also an assumption (pilot wave condition).

To connect this with Z_Q and the previous derivation of the SE we recall (6.1) and note that $\rho(x,t) = lim_{\delta x\to 0}Z_Q(x,\delta x,t)$ with (**6L**) $\int p^2F(x,p,t)dp = lim_{\delta x\to 0}[-\hbar^2(\partial^2 Z_Q(x,\delta x,t)/\partial(\delta x)^2)]$. Then the right side of (6.16) can be written as

(6.25)
$$\int [p - p(x,t)]^2 F(x,p,t)dp = \lim_{\delta x\to 0}\left[-\hbar^2\frac{\partial^2 Z_Q(x,\delta x,t)}{\partial(\delta x)^2} + \frac{\hbar^2}{Z_Q}\left(\frac{\partial Z_Q}{\partial(\delta x}\right)^2\right]$$

This can be rearranged as

$$\overline{(\delta p)^2}\rho(x,t) = -\hbar^2 lim_{\delta x\to 0}Z_Q(x,\delta x,t)\frac{\partial^2 log(Z_Q)}{\partial(\delta x)^2} \tag{6.26}$$

It remains now to give the explicit appearance of this expression and show that it is equivalent to (6.22). Note that Z_Q can be written as

$$Z_Q = \int F(x,p,t)dp + \frac{i\delta x}{\hbar}\int pFdp - \frac{(\delta x)^2}{2\hbar^2}\int p^2Fdp + o((\delta x)^3) \tag{6.27}$$

and this is equivalent (using (**6D**) and (**6E**)) to

$$Z_Q = \rho(x,t) + \frac{i\delta x}{\hbar}p(x,t)\rho(x,t) - \frac{(\delta x)^2}{2\hbar^2}M_2(x,t) + o((\delta x)^3) \tag{6.28}$$

The left side has to be written as $Z_Q = \psi^*(x-(\delta x/2))\psi(x+(\delta x/2))$ and using $\psi = Rexp(iS/\hbar)$ one finds (up to second order in the infinitesimal parameter)

$$Z_Q = \left\{R^2 + \left(\frac{\delta x}{2}\right)^2\left[RR_{xx} - (R_x)^2\right]\right\}exp\left(\frac{i}{\hbar}S_x\delta x\right) \tag{6.29}$$

Explicitly this is

$$Z_Q = R^2 + \frac{i\delta x}{\hbar}R^2 S_x + \frac{(\delta x)^2}{2}\left[\frac{1}{4}R^2\frac{\partial^2 log(R^2}{\partial x^2} - \frac{R^2}{\hbar^2}(S_x)^2\right] \tag{6.30}$$

Comparison with (6.28) gives (**6M**) $\rho(x,t) = R^2(x,t)$ and $p(x,t) = S_x(x,t)$ and (**6N**) $M_2(x,t) = -(\hbar^2/4)\rho(x,t)\partial^2 log(\rho(x,t)) + p(x,t)^2\rho(x,t)$. Using (6.28) one can write then

$$Z_Q = \rho(x,t)\left[1 + \frac{i\delta x}{\hbar}p(x,t) + \frac{(\delta x)^2}{2}\left[\frac{1}{4}\partial_x^2 log(\rho(x,t)) - \frac{p^2(x,t)}{\hbar^2}\right)\right] \tag{6.31}$$

which means that

$$lim_{\delta x\to 0}\frac{\partial^2}{\partial(\delta x)^2}log(Z_Q(x,\delta x,t) = \frac{1}{4}\partial_x^2 log(\rho(x,t)) \tag{6.32}$$

where one has expanded the logarithm in (6.31) up to the second order in δx. This last result gives then, using (6.26),

$$\overline{(\delta p)^2}\rho(x,t) = -\frac{\hbar^2}{4}\rho(x,t)\partial_x^2 log(\rho(x,t)) \tag{6.33}$$

which is equivalent to (6.22) as desired (this comes directly from (**6N**) and (6.17)). Another way of comparing the two approaches is to simply substitute (6.28) in the equation satisfied by the characteristic function which is (via (6.3)) (**6O**) $-i\hbar\partial_t Z_Q - (\hbar^2/m)(\partial^2 Z_Q/\partial x\partial(\delta x)) + \delta x(V_x)Z_Q = 0$. Taking the real and imaginary parts gives then

$$\partial_t\rho(x,t) + \partial_x\left[\frac{p(x,t)}{m}\rho(x,t)\right] = 0; \tag{6.34}$$

$$\partial_t[\rho(x,t)p(x,t)] + \frac{1}{m}\partial_x M_2(x,t) + V_x\rho(x,t) = 0$$

which are (6.14) and (6.15). Thus the restriction (**6K**) is equivalent to postulating the adequacy of the Wigner-Moyal infinitesimal transformation together with the restriction of writing $Z_Q = \psi^*(x-(\delta x/2),t)\psi(x+(\delta x/2),t)$.

7. THE UNCERTAINTY PRINCIPLE

We go now to [13, 21, 36, 52, 61, 62, 63, 76, 86, 90, 96, 97, 103, 118] for some results involving fluctuations and the uncertainty principle. In particular [13] exhibits some fascinating relations between the Heisenberg uncertainty principle and the exact uncertainty principle of Hall-Reginatto and we sketch some of this here. The title of [13] is a question "Is QM based on an uncertainty principle" and we indicate how this is answered in [13] in terms of the quantum potential (QP). One modifies the classical mechanical definition of momentum uncertainty in order to satisfy certain transformation rules. This involves adding a new term to the classical

quadratic momentum uncertainty which has to be proportional to the inverse of a measure of the quadratic position uncertainty. Then one imposes the Hall-Reginatto conditions of causality and additivity of kinetic energy which leads to a complete specification of the functional dependence of the new term requiring it to be essentially the QP (often associated with the idea of quantization - modulo some arguments at times). An observer is now characterized by parameters denoting the statistical position and momentum uncertainties Δx and Δp of its instruments. For example Δx^2 could be the trace of the covariance matrix associated to a given position probability density $\rho(x)$ or as the inverse of the trace of the Fisher matrix associated to ρ. The main postulate here is that under dilatations of space coordinates the parameters Δx and Δp must transform in such a manner that the relation (**7A**) $\Delta x\Delta p\geq(\hbar/2$ is kept invariant. The transformations allowed here are

$$\Delta x'^2=e^{-\alpha}\Delta x^2;\ \Delta p'^2=e^{-\alpha}\Delta p^2+\frac{\hbar^2}{4}(e^{\alpha}-e^{-\alpha})\frac{1}{\Delta x^2}\tag{7.1}$$

where α is real and one sees that such transformations generate a group. Multiplying the terms together in (7.1) gives

$$\Delta x'^2\Delta p'^2=e^{-2\alpha}\Delta x^2\Delta p^2+\frac{\hbar^2}{4}(1-e^{-2\alpha})\tag{7.2}$$

One notes that for $\alpha\to\infty$ one has $\Delta x'^2\Delta p'^2\to(\hbar^2/4)$, if $\Delta x^2\Delta p^2=\hbar^2/4$ then it remains so, and for $\alpha\to-\infty$ one has $\Delta x'^2\Delta p'^2\to\infty$ for any value of $\Delta x^2\Delta p^2\geq(\hbar^2/4)$. Uniqueness of such transformations has not been established. There is some analogy here to special relativity and this is discussed in [13]. In any case one shows now that the stipulations (7.1) impose a radical modification of the laws of dynamics that corresponds precisely to that required in the passage from classical to quantum mechanics. Thus consider $\rho(x)$ and $S(x)$ as basic variables (fields) and specify that the time evolution of any functional via (**7B**) $\mathfrak{A}=\int d^3xF(x,\rho,\nabla\rho,\nabla^2\rho,\cdots,S,\nabla S,\nabla^2S,\cdots)$ is given by (**7C**) $\partial_t\mathfrak{A}=\{\mathfrak{A},\mathfrak{H}_{cl}\}$ where $\mathfrak{H}_{cl}=\int d^3x(\rho|\nabla S|^2/2m)$ and

$$\{\mathfrak{A},\mathfrak{B}\}=\int d^3x\left[\frac{\delta\mathfrak{A}}{\delta\rho(x)}\frac{\delta\mathfrak{B}}{\delta S(x)}-\frac{\delta\mathfrak{B}}{\delta\rho(x)}\frac{\delta\mathfrak{A}}{\delta S(x)}\right]\tag{7.3}$$

The Poisson bracket provides a Lie algebra structure. Applying (**7C**) to ρ and S yields

$$\partial_t\rho=-\nabla\cdot\left(\rho\frac{\nabla S}{m}\right);\ \partial_tS=-\frac{|\nabla S|^2}{2m}\tag{7.4}$$

(note $\nabla S\sim p$ which here is the classical momentum). Now consider the group of space dilatations $x\to exp(-\alpha/2)x$ and its action on ρ and S via (**7D**) $\rho'(x)=exp(3\alpha/2)\rho(exp(\alpha/2)x)$ and $S'(x)=exp(-\alpha)S(exp(\alpha/2)x)$ (α real). Such transformations preserve the normalization of ρ and keep

the dynamical equations (7.3) invariant. Assume now that the average momentum of the particle is vanishing (i.e. a comoving frame is chosen which does not reduce the generality of the results). Then the classical definition of the scalar quadratic momentum uncertainty is given via (**7E**) $\Delta p_{cl}^2 = \int d^3x\rho|\nabla S|^2 = 2m\mathfrak{H}_{cl}$ and under (**7D**) this becomes (**7F**) $\Delta \rho_{cl}^{'2} = exp(-\alpha)\Delta p_{cl}^2$. Also any definition of the scalar quadratic position uncertainty measuring the dispersion Δx^2 of $\rho(x)$ transforms as (**7G**) $\Delta x^{'2} = exp(-\alpha)\Delta x^2$. The requirement of (7.1) to be fundamental now requires one to modify (**7E**) in order to get a quantity whose variance satisfies (7.1). Some argument shows that adding a supplementary term proportional to $\hbar^2$ is needed in order to have Δp^2 transform reasonably under (**7D**). This is accomplished via

$$\Delta p_q^2 = \int d^3x\rho(x)|\nabla S(x)|^2 + \hbar^2\mathfrak{Q} \tag{7.5}$$

where $\mathfrak{Q}$ is to be determined. Applying (**7D**) to (7.4) leads to (**7H**) $\Delta p_q^{'2} = e^{-\alpha}\Delta p_{cl}^2 + \hbar^2(\mathfrak{Q}' - e^{-\alpha}\mathfrak{Q})$. Adding and subtracting $exp(\alpha)\hbar^2\mathfrak{Q}$ yields and equation (**7I**) $\Delta p_q^{'2} = exp(-\alpha)\Delta p_q^2 + \hbar^2(\mathfrak{Q}' - exp(-\alpha)\mathfrak{Q})$. Identifying this with (7.1) requires (**7J**) $\mathfrak{Q}' - exp(-\alpha)\mathfrak{Q} = (1/4\Delta x^2)(exp(\alpha) - exp(-\alpha))$ which, using (7.1) becomes (**7K**) $\mathfrak{Q}' - (1/4\Delta x^{'2}) = exp(-\alpha)[\mathfrak{Q} - (1/4\Delta x^2)]$. There are an infinity of solutions but the form indicates a relation between $\mathfrak{Q}$ and Δx^2 that is scale independent, namely (**7L**) $\mathfrak{Q} = (1/4\Delta x^2)$. This is the only solution for which the relation between Δp_q^2 and Δx^2 is independent of the scale α. Thus this is the needed term to obtain a definition of Δp_q^2 compatible with (7.1). Since Δx^2 depends only on $\rho(x)$ we see that $\mathfrak{Q}$ is a functional of the form (**7B**) that does not depend on S. For the precise form of Δx^2 one can now refer to the Hall-Reginatto results (cf. [21] for a survey and a discussion of relations to Fisher information) which work with entirely different arguments in discovering the additive term $\hbar^2\mathfrak{Q}$. At this point their additional requirements of causality and additivity can be invoked point to obtain $\mathfrak{H}_q = \Delta p_q^2/2m$ and $\mathfrak{Q} = \beta\int d^3x|\nabla\rho(x)^{1/2}|^2$ (with $\beta = 1$) and finally

$$\mathfrak{H}_q = \int d^3x\left[\frac{\rho(x)|\nabla S|^2}{2m} + \frac{\hbar^2}{2m}|\nabla\rho(x)^{1/2}|^2\right] \tag{7.6}$$

The HJ equation $\partial_t S = -(|\nabla S|^2/2m) + (\hbar^2/2m)(\nabla^2\rho(x)^{1/2}/\rho^{1/2}(x)$ is given via (**7C**) along with the continuity equation in (7.3) leading to the standard SE for $\psi = \rho^{1/2}exp(iS/\hbar)$.

One looks next at dilatations under (**7D**) to get

$$\mathfrak{H}_q' = Cosh(\alpha)\int d^3x\left[\frac{\rho|\nabla S|^2}{2m} + \frac{\hbar^2}{2m}|\nabla\rho^{1/2}|^2\right] - \tag{7.7}$$

$$-Sinh(\alpha)\int d^3x\left[\frac{\rho|\nabla S|^2}{2m}-\frac{\hbar^2}{2m}|\nabla\rho^{1/2}|^2\right]$$

where (**7M**) $\mathfrak{H}'_q[\rho,S]=\mathfrak{H}_q[\rho',S']$. One writes then

$$\mathfrak{K}_q=\int d^3x\left[\frac{\rho|\nabla S|^2}{2m}-\frac{\hbar^2}{2m}|\nabla\rho^{1/2}|^2\right]\tag{7.8}$$

In more compact notation now write (**7N**) $\mathfrak{H}'_q=Cosh(\alpha)\mathfrak{H}_q-Sinh(\alpha)\mathfrak{K}_q$ with $\mathfrak{K}'_q=-Sinh(\alpha)\mathfrak{H}_q+Cosh(\alpha)\mathfrak{K}_q$ so that under (**7D**) $(\mathfrak{H}_q,\mathfrak{K}_q)$ transforms as a 2-D Minkowski vector under a Lorentz like transformation. This corresponds to (**7O**) $t'=Cosh(\alpha)t+Sinh(\alpha)\tau$ and $\tau'=Sinh(\alpha)t+Cosh(\alpha)\tau$ and one can write (**7P**) $\partial_t\mathfrak{A}=\{\mathfrak{A},\mathfrak{H}_q\}$ with $\partial_\tau\mathfrak{A}=\{\mathfrak{A},\mathfrak{K}_q\}$; further this transforms via (**7D**) to (**7Q**) $\partial_{t'}\mathfrak{A}'=\{\mathfrak{A}',\mathfrak{H}'_q\}$ and $\partial_{\tau'}\mathfrak{A}'=\{\mathfrak{A}',\mathfrak{K}'_q\}$. Thus this is all covariant under (**7D**). The SE is a particular case of (**7P**) where

$$\mathfrak{A}=\psi=\rho^{1/2}e^{iS/\hbar};\ i\hbar\partial_t\psi=-\frac{\hbar^2}{2m}\nabla^2\psi\tag{7.9}$$

and one obtains also

$$i\hbar\partial_\tau\psi=-\frac{\hbar^2}{2m}\nabla^2\psi+\frac{\hbar^2}{m}\frac{\nabla^2|\psi|}{|\psi|}\tag{7.10}$$

This is an interesting equation to be discussed later (see e.g. [2, 42, 58, 59, 68, 121, 125, 128]). Thus (7.8) and (7.9) are covariant under space dilatations with

$$i\hbar\partial_{t'}\psi'=-\frac{\hbar^2}{2m}\nabla^2\psi';\ i\hbar\partial_{\tau'}\psi'=-\frac{\hbar^2}{2m}\nabla^2\psi'+\frac{\hbar^2}{m}\psi'\frac{\nabla^2|\psi'|}{|\psi'|}\tag{7.11}$$

where (cf. (**7D**))

$$\psi'=e^{3\alpha/4}[\psi(e^{\alpha/2}x)]^{(1/2)(1+exp(-\alpha))}[\psi^*(e^{\alpha/2}x)]^{(1/2)(1-exp(-\alpha))}\tag{7.12}$$

Note that when the transformation of the SE under dilatations is considered ψ transforms as the square root of a density $\rho^{1/2}$ (cf. also [14, 15, 53, 54, 55, 70]). Finally one notes that defining $\mathfrak{s}=\int d^3x\rho(x)S(x)$ (ensemble average of the quantum phase up to a factor of $\hbar$) one has

$$\{\mathfrak{s},\mathfrak{H}_q\}=\int d^3x\left[\frac{\rho|\nabla S|^2|}{2m}-\frac{\hbar^2}{2m}|\nabla\rho^{1/2}|^2\right]=\mathfrak{K}_q\tag{7.13}$$

$$\{\mathfrak{s},\mathfrak{K}_q\}=\int d^3x\left[\frac{\rho|\nabla S|^2}{2m}+\frac{\hbar^2}{2m}|\nabla\rho^{1/2}|^2\right]=\mathfrak{H}_q$$

This leads to a general statement $\mathfrak{A}'=\mathfrak{A}+\delta\alpha\{\mathfrak{A},\mathfrak{s}\}$

REMARK 7.1. Let us gather together some of the ideas connecting [13, 61, 62, 63, 96]. Thus in [13] (for 1-D here)

$$\Delta x^2 \sim \frac{1}{\int[(\rho')2/\rho]dx} = \frac{1}{F} \tag{7.14}$$

and the particle mean square deviation (or variance) is σ_x^2 with $\sigma_x^2 \geq \Delta x^2$. To check this one can follow [49] and write for suitable estimators $\hat{\theta}$ of θ (note $\rho_\theta = \partial_\theta \rho = \rho\partial_\theta log(\rho)$ and $\hat{\theta} = \hat{\theta}(y)$)

$$0 =< \hat{\theta} - \theta >= \int dy\rho(y|\theta)[\hat{\theta}(y) - \theta] \Rightarrow 0 = \int dy\rho_\theta[\hat{\theta} - \theta] - \int \rho dy \Rightarrow \tag{7.15}$$

$$\Rightarrow 1 = \int dy\rho_\theta[\hat{\theta} - \theta] = \int dy\sqrt{\rho}[\hat{\theta} - \theta]\sqrt{\rho}\partial_\theta log(\rho) \leq \int \rho|\hat{\theta} - \theta|^2 \int \rho|\partial_\theta log(\rho)|^2$$

This says that $1 \leq \sigma_x^2 F$ with σ_x^2 the classical variance and $F \sim$ Fisher information. Now recall from #5 before Example 5.1 that it is natural to think of momentum fluctuations in the form $\delta p \sim \rho'/\rho = \partial_x log(\rho) \sim cu$ where u is an osmotic velocity. Then look at the exact uncertainty principle of [61, 62, 63] where one writes $\delta x \Delta p_{nc} = \hbar/2$ (recall $p_{nc} \sim \nabla S + \delta p$ with $\delta p \sim \hat{c}\partial log(\rho)$. In [89] on the other hand there is a crucial condition (**6K**) $\overline{(\delta p)^2 (\delta x)^2} = \hbar^2/4$ which is imposed in the context of a statistical equilibrium situation involving Boltzman type entropy. Here $\overline{(\delta p)^2}$ is determined via ((**6K**) and (6.21) in the form (**7R**) $\overline{(\delta x)^2} = 1/2\gamma$ where $2\gamma = (1/k_B)\partial^2 \mathfrak{S}_{eq}/\partial x^2$ leading to (6.22) where (**7S**) $\overline{(\delta p)^2} \sim -(\hbar^2/4)\partial_x^2 log(\rho)$. Since $\partial^2 log(\rho) \sim \partial(\rho'/\rho) = (\rho''/\rho - (\rho'/\rho)^2$ one sees that for $\rho' = \rho = 0$ outside of a compact Ω (**7T**) $\int \rho\partial^2 log(\rho) = -\int(\rho')^2/\rho = -\int \rho(\rho'/\rho)^2$ and such quantum perturbations to $\nabla S)^2$ would involve the same action as perturbations $(\rho'/\rho)^2$ (corresponding to $\int \rho Q$ where $Q \sim (\partial^2\sqrt{\rho})/\sqrt{\rho} = (1/2)[(\rho''/\rho) - (\rho'^2/2\rho^2]$. ■

We go next to [103] and sketch some of the results. In "geometric quantum mechanics" (GQM) geometry is not prescribed but is determined by the space matter content. Further in [103] one assumes that the affine connections are responsible for quantum phenomena. Given that GQM deals with a Gibbs ensemble of particles, rather than a single particle, one can treat it as a classical theory. Wave equations and the operational prescriptions of standard QM are devoid of physical meaning and are regarded as clever devices to overcome the difficulties inherent in a nontrivial geometric structure (see [21, 102] and Section 2 for more on this). The object in [103] is to show how the Heisenberg uncertainty principle arises in the context of GQM. Note that Planck's constant is conspicuously absent in the treatment. Thus in GQM one assumes that in a parallel displacement $d\mathbf{r}$ the length $\ell = (\mathbf{A}\cdot\mathbf{A})^{1/2}$ of a vector $\mathbf{A}$ changes by an amount (•) $\delta\ell = \ell(\vec{\phi}\cdot d\mathbf{r})$

where $\vec{\phi}$ is the Weyl gauge vector. In nonrelativistic mechanics the spatial metric is Euclidean so the dot means the standard scalar product and we have a Weyl space which generally will have a nonzero scalar curvature $\mathcal{R}$ even if its metric has zero curvature. In GQM the law (•) and hence the Weyl-Ricci curvature is determined by the motion of the particle itself and in turn it acts as a guidance field for the particle (this is perhaps another way to look at Bohmian mechanics). Following [102] (cf. also Section 2) the length transference law may be obtained from a minimum average curvature principle (••) $E[\mathcal{R}(\mathbf{r}(t),t)] = min$ where E denotes the ensemble expectation value. When the space is flat ($\dot{R}=0$) one has from (2.3) (cf. [21, 102]) (♦) $\mathcal{R} = 2|\vec{\phi}|^2 - 4\nabla\cdot\vec{\phi}$. Now let ρ be the probability density of the particle position (properly normalized and vanishing at infinity) and note that

$$(7.16)\quad E[\nabla\cdot\vec{\phi}] = \int_{\mathbf{R}^3}\rho\nabla\cdot\vec{\phi}d^3\mathbf{r} = -\int_{\mathbf{R}^3}\rho\vec{\phi}\cdot\nabla(log(\rho)d^3\mathbf{r} = -E[\vec{\phi}\cdot\nabla log(\rho)]$$

Consequently

$$(7.17)\quad E[\mathcal{R}] = 2E[|\vec{\phi}|^2+2\vec{\phi}\cdot\nabla log(\rho)] = 2E[(\vec{\phi}+\nabla log(\rho)^2] - 2E[(\nabla log(\phi)^2]$$

The minimizing gauge vector ϕ is found from this to be (♦♦) $\vec{\phi} = -\nabla log(\rho)$ and the minimum average scalar curvature is (★) $E[\mathcal{R}] = -2E[|\vec{\phi}|^2]$. Note that the minimal average curvature is negative and moreover from (♦♦) one obtains

(7.18)

$$E[\vec{\phi}(\mathbf{r},t),t)] = \int_{\mathbf{R}^3}\rho(\mathbf{r},t)\vec{\phi}(\mathbf{r},t)d^3\mathbf{r} = -\int_{\mathbf{R}^3}\rho\nabla log(\rho)d^3\mathbf{r} = -\oint_S \rho\mathbf{n}dS = 0$$

where S is a closed surface at ∞ enclosing $\mathbf{R}^3$ (where $\rho = 0$). One notes also that the mean square deviation for $|\vec{\phi}|$ is not zero in general. In fact from (★) and (7.18) one finds the mean square deviation via

$$(7.19)\qquad \Delta|\vec{\phi}| = \{E[|\vec{\phi}|^2] - E^2[\vec{\phi}]\}^{1/2}] = 2^{-1/2}[E(-\mathcal{R}]^{1/2}$$

Now let Δq be the root mean square deviation of the particle position so that (7.18) and the Schwarz inequality give

$$(7.20)\qquad \Delta q\Delta|\vec{\phi}| \geq Cov\{(\mathbf{r}-E[\mathbf{r}])\cdot(\vec{\phi}-E[\vec{\phi}])\} = E[\mathbf{r}\cdot\vec{\phi}]$$

where Cov denotes covariance. The average on the right side of (7.20) can be computed using (♦♦) and one gets (using (7.19))

$$(7.21)\qquad \Delta q\Delta|\vec{\phi}|^2 \geq \int_{\mathbf{R}^3}\rho\nabla\cdot\mathbf{r}d^3\mathbf{r} = 3 \Rightarrow \Delta q(E[-\mathcal{R}])^{1/2} \geq \frac{3}{\sqrt{2}}$$

This is the fundamental relation between the particle root mean square deviation and the average space curvature. The space may be flat on average only if the particle is completely delocalized - particle localization forces

the space to be curved. One notes that (7.21) is purely geometrical and Planck's constant is not involved. However (7.21) does imply the Heisenberg uncertainty principle provided the prescriptions of standard QM are used. Thus for $\psi = \sqrt{\rho}exp(iS/\hbar)$ with a particle of mass m and probability density $\rho = |\psi|^2$ one has trajectories (★★) $d\mathbf{r}/dt = \nabla S(\mathbf{r}, t)/m$ where

$$S_t + \frac{|\nabla S|^2}{2m} + V - \frac{\hbar^2}{16m}\mathcal{R} = 0 \tag{7.22}$$

(cf. Section 2 and [102]). The averages obtained using the operator methods of standard QM do not coincide in general with the averages made on the classical ensemble. Thus using $<\ ,\ >$ for QM and $E[\ ]$ for ensemble averages one gets (using (7.22))

$$<\mathbf{r}>= E[\mathbf{r}];\ <\mathbf{p}>= E[\nabla S];\ <\hat{p}^2>= E[|\nabla S|^2] + \frac{\hbar^2}{8}E[-\mathcal{R}] \tag{7.23}$$

where $-i\hbar\nabla \sim \hat{p}$. Note here also that defining $\Delta p = [<\mathbf{p}- <\mathbf{p}>^2]^{1/2}$ it follows immediately via (7.23) that

$$(\Delta p)^2 = E[(\nabla S - E[|\nabla S|)^2] + \frac{\hbar^2}{8}E[-\mathcal{R}] \Rightarrow \Delta p \geq \frac{\hbar}{2\sqrt{2}}(E[-\mathcal{R}])^{1/2} \tag{7.24}$$

leading to the Heisenberg type inequality (**7U**) $\Delta p\delta Q \geq (3/2)\hbar$ via (7.21).

REMARK 7.2. The relation $\Delta x\Delta p \geq (\hbar/2)$ can be traced in an analogous manner using $\Delta x\Delta\phi_x \geq E[x\phi_x]$ in place of (7.20).[■

Finally one notes that in (7.24) there is a random motion part and a curvature part and it is the curvature part that forces Heisenberg's position momentum uncertainty relation. However this contribution is a mere consequence of the operator formulation of QM and, unlike (7.21), it has no physical meaning in a physically consistent approach to GQM. In fact, GQM being a classical theory, the particle momentum should be defined as $p = \nabla S$ and the mean square deviation should be given by the first term on the right in (7.24).

REMARK 7.3. In reading over the second paper in [118] one is struck by similarities to the approach of Nottale (see [21, 35, 86]). The QP term arises basically because the quantum paths are not differentiable (à la Feynman they have fractal dimension 2) and as indicated in [21] this corresponds to expressing the QP in terms of an osmotic velocity $\mathbf{u}$ (cf. #1 before (5.2) in Section 5). In the Nottale approach $\mathbf{u}$ only appears because of "jagged" paths and in the presence of smooth (i.e. differentiable) paths the QP is zero and there is no SE. If there are nonsmooth paths then the fact that $\mathbf{u} = D\nabla log(\rho)$ (with $\vec{\phi} = -\nabla log(\rho)$) suggests that $\mathbf{u}$ is a curvature phenomena but that is misleading - it is really a diffusion phenomena and for

D=0 **u** will not appear. In more detail, following Nottale, one can write

$$\rho_t + div(\rho b_+) = D\Delta\rho;\ \rho_t + div(\rho b_-) = -D\Delta\rho \tag{7.25}$$

But $b_\pm$ represents e.g. right and left derivatives at a given point so smooth paths give $b_+ = b_-$ and hence $\mathbf{u} \sim (1/2)(b_+ - b_-) = 0$ with $D = 0$ necessarily via $D = -D$. Hence smooth paths imply no Wiener process and no SE. One can still have curvature via a nonzero Weyl vector $\phi \sim -\nabla log(\rho)$. The SE and QM arise via an assumption of diffusion and nonsmooth paths which is equivalent to introducing a nonzero QP. This does not answer the question "Why QM?" but does seem to provide evidence that (**7U**) is the fundamental GQM idea and the Heisenberg inequality arises only after introducing diffusion with a particular diffusion coefficient $\hbar/2m$ (cf. [42]), or introducing a Schrödinger wave function containing $\hbar$, or perhaps after introducing all the miracle machinery of Hilbert space QM, etc. This is all consistent with comments made in Remarks 2.1 and 2.2. The factor $\hbar$ is "gratuitous" and is introduced in QM either via the wave function idea $\psi = Rexp(iS/\hbar)$ or via a diffusion coefficient $D = \hbar/2m$. ∎

8. GEOMETRY AND QUANTUM MATTER

We have seen already a number of striking relations between geometry and quantum matter some of which we repeat here via equations. Thus

(1) From (2.9) we have

$$Q \sim -\frac{\hbar^2}{16m}\left[\dot{\mathcal{R}} + 2\left\{\phi_i\phi^i - \frac{2}{\sqrt{g}}\partial_i(\sqrt{g}\phi^i)\right\}\right] \tag{8.1}$$

(2) From (2.10) follows

$$\phi_k\phi^k - 2\partial_k\phi^k \sim -\left(\frac{|\nabla\rho|^2}{\rho^2} - \frac{2\Delta\rho}{\rho}\right) = 4\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{8.2}$$

(3) For n=4 from (2.18) results

$$\mathcal{R} = \dot{\mathcal{R}} - 3\left[\frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu + \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\phi_\nu\right] = \mathcal{R} + \mathcal{R}_w \tag{8.3}$$

(4) From (2.3) for n=3 we have

$$\mathcal{R} = \dot{\mathcal{R}} + 2\left[\phi_i\phi^i - 2\left(\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\phi^i)\right)\right] \tag{8.4}$$

(5) Further one knows that

$$\mathfrak{M}^2 = m^2 e^Q;\ Q = \frac{\hbar^2}{m^2c^2}\frac{\Box\sqrt{\rho}}{\sqrt{\rho}} = \frac{\hbar^2}{m^2c^2}\frac{\nabla^2|\psi|}{|\psi|} \tag{8.5}$$

(6) The following equations also relate the Weyl vector $\vec{\phi}$ and the Dirac field β

$$\beta \sim \mathfrak{M};\ \beta_0 \to \beta = \beta_0 e^{-\Xi};\ \phi_\mu \to \phi_\mu + \partial_\mu \Xi \tag{8.6}$$

(7) Further $Q \sim -(m^2/2)\mathbf{u}^2 - (\hbar/2)\partial\mathbf{u}$ ($\mathbf{u} \sim$ osmotic velocity) with

$$\mathbf{u} = D\nabla log(\rho) = \frac{\hbar}{m}\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}};\ \vec{\phi} = -\nabla log(\rho); \tag{8.7}$$

Thus in particular the quantum mass $m_Q \sim \mathfrak{M}$ is given via Q, or via ρ, or via $\vec{\phi}$ so $\vec{\phi}$ determines $\mathfrak{M}$ and conversely $\mathcal{M}$ determines ρ via Q. Recall that $\Delta R + \beta QR = 0$ has unique solutions in say $H_0^1(\Omega)$ where $\beta = -(2m/\hbar^2)$. In fact $\Delta R + \beta QR = \lambda R$ has a unique solution for $\lambda \neq \Sigma$ where Σ is a countable set $\Sigma \subset \mathbf{R}$ (cf. [22]) so if $0 \neq \Sigma$ then Q determines R uniquely. In any event ρ itself determines $\mathfrak{M}$ and $\vec{\phi}$ (as well as $\mathbf{u}$). Since ρ, or $\vec{\phi}$, or β determine the Weyl Dirac geometry (the Riemannian metric is assumed fixed here) one seems to have already an excellent theory for quantum perturbations of a classical Riemannian geometry. Is this not an important desideratum of quantum gravity theory? To ask for quantum perturbations that maintain a Riemannian structure seems perhaps excessive (cf. [28] (although conformal equivalence is established via $\mathfrak{M}^2/m^2$) and we are not trying to answer the questions of basic structure (for this see e.g. [7, 8, 34, 77, 72, 100, 132]).

9. REMARKS ON MANY WORLDS

We refer back to Remark 2.2, Remark 2.5, and Sections 3-5. Consider a 3-D Riemannian manifold M with metric g_{ij} and Riemann curvature $\dot{\mathcal{R}}$. Let m be a mass and consider an ensemble of particles of mass m distributed via a probability density P which generates a mass density $\rho = mP(x,t)$. Write $\hat{\rho} = \rho/\sqrt{g}$ and observe that a natural classical background for quantum evolution in M is provided via a Weyl geometry on M based on a Weyl vector $\vec{\phi} = -\nabla log(\hat{\rho})$. Then the SE (2.6) for $\psi = Rexp(iS/\hbar)$, namely

$$i\hbar\psi_t - \frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}[\partial_i(\sqrt{g}g^{ik}\partial_k)]\psi + [V - \frac{\hbar^2}{16m}\dot{\mathcal{R}}]\psi \tag{9.1}$$

corresponds to classical evolution (cf. (**2G**), (2.4), and (2.5))

$$\partial_t S + \frac{1}{2m}g^{ik}\partial_i S\partial_k S + V - \frac{\hbar^2}{2m}\mathcal{R} = 0; \tag{9.2}$$

$$\partial_t\hat{\rho} + \frac{1}{m\sqrt{g}}\partial_i(\sqrt{g}\partial_i S\hat{\rho}) = 0$$

where

$$\mathcal{R} = \dot{\mathcal{R}} + \frac{8}{\sqrt{g\hat{\rho}}}\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\rho})];\ Q \sim -\frac{\hbar^2}{16m}\mathcal{R} \tag{9.3}$$

Here $\mathcal{R} = \mathcal{R}_w$ is the Ricci-Weyl curvature.

REMARK 9.1. Thus assume M is Euclidean with $\dot{\mathcal{R}} = 0$ for simplicity so $g \sim 1$ and $\hat{\rho} = \rho$. The Weyl vector is then $\vec{\phi} = -\nabla log(\rho)$ and

$$\mathcal{R} = \frac{8\Delta\sqrt{\rho}}{\sqrt{\rho}};\ Q \sim -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{9.4}$$

This suggests a kind of many worlds scenario for quantum motion. Thus given a probability distribution $P(x,t)$ with $\rho = mP$ as above one has wave functions $\psi = Rexp(iS/\hbar)$ where $R \sim \sqrt{\rho}$ (and some normalization $\int R^2 dx = \int \rho dx = 1$ for example is imposed). This seems to say that each such wave function $Rexp(iS/\hbar)$ generates a family of time dependent Weyl geometries evolving via (9.2). The choice of $R \sim \sqrt{\rho}$ determines the geometry and we have an infinite number of Weyl space scenarios, each created by a choice of probability distribution $P(x,t)$. Thus a quantum particle is associated to an infinite number of time dependent Weyl space paths, each determined by some particular wave function $\psi = Rexp(iS/\hbar)$ where $R = |\psi|$ can be prescribed via a suitably normalized probability distribution $P(x,t)$. In view of (9.4) each ρ as above is associated to a quantum potential Q and if the association $R \leftrightarrow Q$ holds (as in Remark 5.1) then it could be said that Q determines the Weyl spaces. If $P(x,t) = P(x)$ is independent of time the Weyl space is fixed. ■

REMARK 9.2. One can rephrase the above as follows. Take M as in Remark 9.1 with $\dot{\mathcal{R}} = 0$ and posit a SE (9.1) in the form (**9A**) $i\hbar\psi_t = -(\hbar^2/2m)\Delta\psi + V\psi$. Every wave function $\psi = Rexp(iS/\hbar)$ gives rise to a probability distribution $R^2 = |\psi|^2 = \rho$ and thence to a time dependent family of Weyl geometries via $\vec{\phi} = \nabla log(\rho)$ within which R and S evolve classically via (9.2) which we rewrite here as

$$\partial_t S + \frac{1}{2m}S_x^2 + V + Q = 0;\ \partial_t \rho + \frac{1}{m}(\rho S_x)_x = 0;\ Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{9.5}$$

Thus every wave function picks some worlds (families of Weyl spaces) describing the evolution of R and S. Recalling that $Q \sim -(\hbar^2/16m)\mathcal{R}_w$ one could also say that the quantum potential (defined via ρ) determines the family of Weyl spaces and indicates a quantization of the classical evolution (9.5) (with $Q = 0$). Here $\hbar$ and m arise via the SE and one could consider the original situation $\rho = mP = R^2$ here. If one expects $\int R^2 dx = 1$, which means $m\int Pdx = 1$ so that the choice of P is constrained via $\int Pdx = 1/m$, then it seems necessary to take $P = P(x)$ with a fixed Weyl space; in this situation SE corresponds to an infinite number of classical systems evolving in a fixed Weyl space, each determined by a wave function of the SE with $S = S(\rho, V)$. ■

REMARK 9.3. It is not at all clear whether or not all this has any relation to the many worlds interpretation (MWI) (cf. [16, 17, 69, 101, 120, 123, 124, 127, 130]) and we will not belabor the idea. In many respects Bohmian mechanics as well as the consistent (or decoherent) histories approach (although different) seem to give a deeper insight into quantum processes than does the standard Copenhagen framework. Both are based on trajectories but represent a given history by different operators. We refer here to [19, 51, 57, 64, 66, 67, 71] for more on this. The important relation between quantum mechanics (QM) and mathematical statistics elucidated in [11] will be discussed below and this is relevant to any theory of wave functions. ∎

Let us now examine the situation in Remarks 9.1 and 9.2 in a more coherent manner, based on the Ricci-Weyl curvature $\mathcal{R} = \mathcal{R}_w \sim Q$ as the basic ingredient (cf. (9.3) - (9.4)). Thus let Ξ be a class of quantum potentials, $Q = Q(x,t)$, for which $\Delta R + \beta QR = 0$ ($\beta > 0$) has a unique solutiion $R \in H_0^1(\Omega)$ (where eventually $\beta \sim 2m/\hbar^2$). The t dependence in R arises directly from Q in this context. We specify now a Ricci-Weyl curvature (**9A**) $\mathcal{R}_w(x,t) = -(16m/\hbar^2)Q(x,t) = -8\beta Q(x,t)$ and think of $R(x,t) = R \sim \sqrt{\rho}$. We assume in the background a SE with potential V and via (9.5) one has e.g.

$$\frac{1}{m}(\rho S_x)(x,t) = -\int_0^x \partial_t \rho dx + F(t) \tag{9.6}$$

where $(1/m)(\rho S_x)(0,t) = F(t)$ and $\rho(0,t)$ is known. If we are dealing with particle motion where $S_x \sim p$ then $S_x(0,t) \sim p(0,t)$ and we see also that S_t is known from (9.5) up to a factor $p(0,t)$. Hence given $\mathcal{R}_w \sim -8\beta Q$ based on a SE with potential V we can determine S_x and S_t up to a factor $p(0,t) = S_x(0,t)$ leading to

THEOREM 9.1. Given a SE based on a potential V in a region $\Omega \subset \mathbf{R}^3$ let $\mathcal{R}_w \sim -8\beta Q$ be given with $Q \in \Xi$ ($\beta \sim 2m/\hbar^2$). Then $\mathcal{R}_w$ can be associated with a wave function $\psi = Rexp(iS/\hbar)$ and a Weyl geometry based on a Weyl vector $\vec{\phi} = -\nabla log(\rho)$ where $R = \sqrt{\rho}$ is completely determined with S_x and S_t known up to a factor $S_x(0,t) \sim p(0,t) \sim F(t)$. Thus for $Q \in \Xi$ the quantum potential, or equivalently the Weyl-Ricci curvature, determines (modulo $F(t)$ and a constant) a Weyl space path describing the evolution of R and S with corresponding wave function ψ.

Suppose for a given $\mathcal{R}_w \sim -8\beta Q$ and its corresponding $R(x,t)$ one had different $S(x,t)$, e.g. S and $\hat{S}$ with $S_x \sim F(t)$ and $\hat{S}_x \sim \hat{F}(t)$ so that from (9.5) and (9.6)

$$\frac{1}{m}\rho(\hat{S}_x \mp S_x) = \hat{F} \mp F = G_{\mp}(t) \Rightarrow \hat{S}_x \mp S_x = \frac{mG_{\mp}(t)}{\rho(x,t)} \tag{9.7}$$

$$(9.8)\quad \hat{S}_t - S_t = -\frac{1}{2m}(\hat{S}_x^2 - S_x^2) = -\frac{1}{2m}(\hat{S}_x - S_x)(\hat{S}_x + S_x) = -\frac{m}{2\rho^2}(G_- G_+)$$

COROLLARY 9.1. Given a SE based on a potential V, a wave function ψ determines a unique Weyl space path and Weyl-Ricci curvatures $\mathcal{R}_w$ with corresponding evolution of R and S. On the other hand a given Weyl-Ricci curvature $\mathcal{R}_w \sim -8\beta Q$ with $Q \in \Xi$ determines a unique Weyl space path for motion of R, S with R known completely and S determined modulo (9.7)-(9.8).

COROLLARY 9.2. As stated in Remark 9.2 a given SE is associated to a collection of Weyl space paths (determined via wave functions) describing the evolution of R and S. The Weyl space path is completely characterized via $Q \in \Xi$ (i.e. by Weyl-Ricci curvatures).

9.1. **QUANTUM INFORMATICS.** There is a fascinating series of papers indicated in [11] and we only sketch here a few ideas. Let $\psi(x) \in L^2(\mathbf{R})$ be given and consider

$$(9.9)\qquad \psi(x) = \frac{1}{\sqrt{2\pi}}\int \tilde{\psi}(p)e^{ipx}dp;\ \tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}}\int \psi(x)e^{-ipx}dx$$

where (naturally) $\int |\psi|^2 dx = \int |\tilde{\psi}|^2 dp$. One writes (**9D**) $P(x) = |\psi(x)|^2$ and $\tilde{P}(p) = |\tilde{\psi}(p)|^2$ with normalization (**9E**) $\int P(x)dx = \int \tilde{P}(p)dp = 1$. The coordinate and momentum probability distributions $P(x)$ and $\tilde{P}(p)$ are called mutually complementary statistical distributions and one notes that e.g. information about the phase S of ψ is lost. For an experimental extracting of information it is not sufficient to use only one fixed representation (Bohr's complementarity principle). There is no distribution $P(x,p)$ corresponding to the $P(x)$ and $\tilde{P}(p)$ distributions since this would violate the Heisenberg uncertainty principle. One shows in [11] that classical statistics is incomplete here while quantum statistics is complete via the introduction of ψ. Thus write

$$(9.10)\qquad P(x) = \psi^*\psi = \frac{1}{2\pi}\int dpdp_1\tilde{\psi}^*(p)\tilde{\psi}(p_1)e^{-ix(p-p_1)} =$$
$$\frac{1}{2\pi}\int dudp\tilde{\psi}^*(p)\tilde{\psi}(p-u)e^{-ixu} = \frac{1}{2\pi}\int f(u)e^{-ixu}du$$

where (**9F**) $f(u) = \int dp\tilde{\psi}^*(p)\tilde{\psi}(p-u) = \int dp\tilde{\psi}^*(p+u)\tilde{\psi}(p)$. Thus one has (**9G**) $P(x) = (1/2\pi)\int f(u)exp(-ixu)du$ with (**9H**) $f(u) = M(exp(iux)) = \int P(x)exp(ixu)dx$ (mean value). Similarly (**9I**) $\tilde{f}(t) = \int \tilde{P}(p)exp(ipt)dp = M(exp(ipt))$ for (**9J**) $\tilde{f}(t) = \int dx\psi^*(x-t)\psi(x)$. This leads to the assertion that in order for the function $f(u)$ to be a characteristic function (as above) it is necessary and sufficient that it be represented as a convolution as in (**9F**) with $\tilde{\psi}$ satisfying $\int dp|\tilde{\psi}(p)|^2 = 1$. To see the necessity let

$f(u)$ be a characteristic function so by (**9G**) it defines a density $P(x)$. Let $\psi(x) = \sqrt{P(x)}exp(iS(x))$ for arbitrary real S (e.g. $S = 0$); this amounts to completing a classical statistical distribution to a quantum state. Then $\tilde{\psi}(p)$ defined via (9.9) provides the decomposition (**9F**) (thus $f(u) \to \tilde{\psi}(p)$). For sufficiency let $f(u)$ be represented via (**9F**) via $\tilde{\psi}(p)$ (normalized as in (**9D**)). Then $f(u)$ will be a characteristic function for $P(x)$ defined via (**9G**) (and normalized). Thus $\tilde{\psi}(p) \to f(u) \to P(x)$. We note that even in classical statistics the equation (**9F**) implicitly reveals the existence of a momentum space and the corresponding wave function $\tilde{\psi}(p)$ and suggests the paucity of classical notions of probability. Of course from (**9F**) one cannot derive a unique wave function $\tilde{\psi}(p)$ and (**9D**) does not yield $\psi(x)$ unambiguously. Hence one classical probability distribution may be described by a number of quantum objects; in order for the statistical theory to be "complete" it has to be expanded in a manner to obtain a quantum state vector ψ (e.g. via introduction of a phase multiplier).

One notes that moments of a random variable can be calculated via means of the characteristic function. Thus for $f(u) = \int P(x)exp(ixu)dx$ one has (**9K**) $f^{(k)}(0) = i^k M(x^k)$ $(k = 0, 1, 2, \cdots)$. Simple calculations then lead to representations $\hat{x} \sim i\partial_p$ in momentum space and $\hat{p} = -i\partial_x$ in coordinate space. Consequently e.g. $\hat{p}\hat{x} - \hat{x}\hat{p} = -i$ is invariant under change of representation space. We recall also the standard Cauchy-Schwartz-Bunyakowski inequality (**9L**) $| < \phi|\psi > |^2 \leq < \phi|\phi >< \psi|\psi >$ and one introduces a "fidelity" F via (**9M**) $F = (| < \phi|\psi > |^2/ < \phi|\phi >< \psi|\psi >)$ with $0 \leq F \leq 1$. The Heisenberg uncertainty relation can be derived via consideration of

(9.11)
$$F(\xi) =< \psi|(-i\xi\hat{p}+\hat{x})(i\xi\hat{p}+\hat{x})\psi >= \xi^2 M(\hat{p}^2) - i\xi M(\hat{p}\hat{x} - \hat{x}\hat{p}) + M(\hat{x}^2) \geq 0$$

Setting e.g. (**9N**) $D_p = M(\hat{p}^2) - (M(\hat{p}))^2$ this leads to (**9O**) $D_x D_p \geq 1/4$ with equality only when $(i\xi\hat{p}+\hat{x})|\psi >= 0$ for some ξ (which means Gaussian states). A more general result is the Robertson-Schrödinger uncertainty relation (cf. [11, 99]). Thus let z_1 and z_1 be two observables (centered so $(M(z_1) = M(z_2) = 0)$ and consider

$$F(\xi) =< \psi|(\xi exp(-i\phi)z_2 + z_1)(\xi exp(i\phi)z_2 + z_1)|\psi >\geq 0 \tag{9.12}$$

Here ξ and ϕ are real and one defines the covariance as (**9P**) $cov(z_1, z_2) = (1/2) < \psi|z_1z_2 + z_2z_1|\psi >$. Let $z_1z_2 - z_2z_2 = iC$ where C is Hermitian so that (**9Q**) $M(C) = -i < \psi|z_1z_2 - z_2z_1|\psi >$ leading to

$$F(\xi) = \xi^2 M(z_2^2) + \xi[2cov(z_1, z_2)cos(\phi) - M(C)Sin(\phi)] + M(z_1^2) \tag{9.13}$$

Set $\rho^2 = 4(cov(z_1, z_2))^2 + (M(C))^2$ and choose an angle β so that

$$2cov(z_1, z_2) = \rho Cos(\beta)\ and\ M(C) = \rho Sin(\beta) \Rightarrow \tag{9.14}$$

$$\Rightarrow F(\xi) = \xi^2 M(z_2^2) + \xi\rho Cos(\phi+\beta) + M(z_1^2) \geq 0$$

Now choose ϕ so that $Cos(\phi+\beta) = 1$ which yields

$$M(z_1^2)M(z_2^2) = D(z_1)D(z_2) \geq \frac{\rho^2}{4} = \left((cov(z_1,z_2))^2 + \frac{(M(C))^2}{4}\right) \quad (9.15)$$

Then define a correlation coefficient (**9R**) $r = [cov(z_1,z_2)/\sqrt{D(z_1)D(z_2)}]$ leading to

$$D(z_1)D(z_2) \geq \frac{(M(C))^2K^2}{4};\ K = \frac{1}{\sqrt{1-r^2}} \quad (9.16)$$

Here K is analogous to the Schmidt number (cf. [11], papers 6 and 7 and [43, 75, 95] for details). When $z_1 = x$ and $z_2 = p$, C is unitary, and one obtains (**9S**) $D(x)D(p) \geq (K^2/4)$ leading to $\Delta x\Delta p \geq (K/2)$. We note that since $\hat{x}$ and $\hat{p}$ do not commute their quantum covariance can not be estimated by their sampling as for a classical covariance; for the corresponding estimate one needs the wave function $\psi(x) = \sqrt{\rho(x)}exp(iS(x))$ which leads then to

$$cov(x,p) = \frac{1}{2} < \psi|xp + px|\psi >= \int x\frac{\partial S(x)}{\partial x}\rho(x)dx \quad (9.17)$$

where $\partial_x S = p$ is the momentum. Fisher information and the Cramer-Rao inequality are also discussed in [11]. In summarizing one postulates in [11]

(1) The principle object of quantum informatics is a quantum system. The evolution of the quantum system is described via probability amplitudes which construct state vectors in Hilbert space.
(2) The state vectors can be defined in different equivalent representations and are thus connected by unitary transformations which describe the time evolution of the quantum system.
(3) Measurements made in different unitary interconnected basis representations generate a set of mutually complementary statistical distributions. For a fixed basis the square of the absolute value of the probability amplitude defines the probability of the quantum system detection in a corresponding basis state.
(4) The space for a composite system state is produced by the tensor product of the states of individual systems.

A conclusion reached here is that QM is a root statistical model, based not on the actual probabilities but on their square roots; this is connected to the flow of half-densities under Bohmian flow in phase space as in [53, 54, 55, 70].

References

[1] R. Adler, M. Bazin, and M. Schiffer, Introduction to general relativity, McGraw Hill, 1965
[2] G. Auberson and P. Sabatier, Jour. Math. Phys., 35 (1994), 4028-4040
[3] J. Audretsch, Phys. Rev. D, 27 (1983), 2872-2884
[4] J. Audretsch, F. Gähler, and N. Straumann, Comm. Math. Phys., 95 (1984), 41-51
[5] O. Barabash and Y. Shtanov, astro-[h 9904144; hep-th 9807291
[6] J. Acacio de Barros and N. Pinto-Neto, gr-qc 9611029; Inter. Jour. Mod. Phys. D, 7 (1998), 201
[7] J. Beckenstein, astro-ph 0403694 and 0412652
[8] J. Beckenstein and R. Sanders, astro-ph 0509519
[9] G. Bertodli, A. Faraggi, and M. Matone, hep-th 9909201
[10] A. Blaut and J. Kowalski-Glikman, gr-qc 9506081, 9509040, 9607004, and 9710136
[11] Y. Bogdanov, quant-ph 0303013, 0303014, 0310011, 0312042, 0509213, 0507031, 0512062, 0605208, and 0612025
[12] J. Bolivar, Quantum classical correspondence, Springer, 2004
[13] L. Brenig, quant-ph 0608025 and 0610142
[14] M. Brown and B. Hiley, quant-ph 0005026
[15] M. Brown, quant-ph 9703007 and 0102102
[16] H. Brown and D. Wallace, quant-ph 0403094
[17] H. Brown, E. Sjöqvist, and G. Bacciagaluppi, quant-ph 9811054
[18] C. Brukner and A. Zeilinger, quant-ph 0212084
[19] T. Brun and J. Hartle, Phys. Rev. D, 60 (1999), 123503
[20] A. Budiyono, quant-ph 0512235 and 0601212; physics 0609216
[21] R. Carroll, Fluctuations, information, gravity, and the quantum potential, Springer, 2006
[22] R. Carroll, quant-ph 0506075
[23] R. Carroll, gr-qc 0512146; physics 0511176 and 0602036
[24] R. Carroll, Foundations of Physics, 35 (2005), 131-154
[25] R. Carroll, Proc. Conf. Nonlinear Physics, Gallipolli, Italy, 2006 (to appear)
[26] R. Carroll, Inter. Jour. Evolution Equation, 1 (2005), 23-56; Applicable Analysis, 84 (2005), 1117-1149
[27] R. Carroll, quant-ph 0309023 and 0309159
[28] R. Carroll, physics 0511176 and 0602036; gr-qc 0512046; Metric fluctuations, entropy, and the WDW equation, Proc. Conf. Nonlin. Phys., Gallipoli, Italy, June 2006, to appear in Teor. i Mat. Fizika
[29] R. Carroll, gr-qc 0501045
[30] R. Carroll, On the quantum potential, geometry, and phase space, 2006, Work sheets, 56 pages
[31] R. Carroll, Abstract methods in partial differential equations, Harper-Row, 1969
[32] R. Carroll, Quantum theory, deformation, and integrability, North-Holland, 2000
[33] C. Castro, Foundations of Physics, 22 (1992), 569-615; Foundations of Physics Lett., 4 (1991), 81-99; Jour. Math. Phys., 31 (1990), 2633-2626
[34] C. Castro, On dark energy, Weyl's geometry, and different derivations of the vacuum energy density, Found. Phys., to appeaR
[35] M. Celerier and L. Nottale, quant-ph 0609107 and 0609161; gr-qc 0505012; hep-th 0112213 and 0210027
[36] I. Chakrabarty, quant-ph 0511169
[37] M. Consoli, hep-th 0109215, 0002098, and 0306070; gr-qc 0306105

[38] J. Cresson, math.GM 0211071
[39] L. Crowell, Quantum fluctuations of spacetime, World Scientific, 2005
[40] M. Davidson, Physica A, 96 (1979), 465
[41] D. Delphenich, gr-qc 0211065 and 0209091
[42] H. Doebner, G. Goldin, and P. Natterman, Jour. Math. Phys., 40 (1999), 49-63
[43] J. Eberly, quant-ph 0508019
[44] L. Evans, Partial differential equations, Amer. Math. Soc., 1998
[45] A. Faraggi and M. Matone, Inter. Jour. Mod. Phys. A, 5 (2000), 1869-2017
[46] A. Faraggi and M. Matone, hep-th 9801033, 9705108, and 9809125
[47] A. Faraggi and M. Matone, Phys. Lett. B, 437 (1998), 369; 445 (1999), 77 and 357; 150 (1999), 34; Phys. Lett. A, 249 (1998), 180; Phys. Rev. Lett., 78 (1997), 163-166; hep-th 9809127
[48] E. Floyd, Phys. Rev. D, 26 (1982), 1339-1347; 29 (1984), 1841-1844; 25 (1982), 1547-1551; 34 (1986), 3246-3249; Phys. Lett. A, 259 (1996), 259-263
[49] B. Frieden, Physics from Fisher information, Cambridge Univ. Press, 1999
[50] P. Garbaczewski, cond-mat 0211362 and 0301044; quant-ph 0408192 and 0504096
[51] M. Gell-Mann and J. Hartle, quant-ph 0609190
[52] P. Gibilisco and T. Isola, math-ph 0509046
[53] M. deGosson, math.SG 0504013 and 0411053; The principles of Newtonian and quantum mechanics, Imperial College Press, 2001; Maslov classes, metaplectic representation, and Lagrangian quantization, Wiley-VCH, Berlin, 1997
[54] M. deGosson, Jour. Phys. A, 37 (2004), 7297-7314; quant-ph 0404072; math.SG 0503013; math-ph 0505073
[55] M. deGosson, math.SG 0411453 and 0504013; Jour. Phys. A, 38 (205), L324-L329
[56] W. Graf, gr-qc 0602054; Phys. Rev. D, 67 (2003), 023002 and 38 (2005), L325-L329
[57] R. Griffiths, Phys. Rev. Lett., 70 (1993), 2201-2204; Phys. Rev. A, 54 (1996), 2759-2774 and 57 (1998), 1604-1618; quant-ph 9902059
[58] F. Guerra and R. Marra, Phys. Rev. D, 28 (1983), 1916-1921
[59] F. Guerra and L. Morato, Phys. Rev. D, 27 (1983),1774-1786
[60] V. Guillemin and S. Sternberg, Symplectic techniques in physics, Cambridge Univ. Press, 1984
[61] M. Hall, Gen. Relativ. Gravit., 37 (2005), 1505-1515; Phys. Rev. A, 62 (2000), 012107 (gr-qc 0408098; quant-ph 9912055)
[62] M. Hall, K. Kumar, and M. Reginatto, Jour. Phys. A, 36 (2003), 9779-9794
[63] M. Hall and M. Reginatto, Jour. Phys. A, 35 (2002), 3289-3303; Fortschr. Phys., 50 (2002), 646-651
[64] J. Halliwell, Phys. Rev. Lett., 83 (1999), 2481-2485; Phys. Rev. A, 58 (1998), 105015 and 60 (1999), 105031; quant-ph 0305084, 0507136, and 0607132
[65] I. Hamilton, R. Mosna, and L. Delle Site, physics 060918
[66] J. Hartle, quant-ph 0209104; gr-qc 9404017
[67] J. Hartle, R. Laflamme, and D. Marolf, Phys. Rev. D, 51 (1995), 7007-7016
[68] A. Heslot, Phys. Rev. D, 31 (1985), 1341-1348
[69] C. Hewitt-Horsman, quant-ph 0210204 and 0310014
[70] B. Hiley, in Quo vadis quantum mechanics, Sringer, 2005, pp. 299-324
[71] B. Hiley, R. Callaghan, and O. Maroney, quant-ph 0010020
[72] M. Israelit, The Weyl-Dirac theory and our universe, Nova Science Pub., 1999; Found. Phys., 29 (1999), 1303-1322; 32 (2002), 295-321 and 945-961
[73] C. Kiefer, Quantum gravity, Oxford Univ. Press., 200
[74] J. Kowalski-Glikman, gr-qc 9511014
[75] L. Lamata and J. Leon, quant-ph 0410167

[76] S. Luo, Jour. Phys. A, 35 (2002), 5181-5187
[77] P. Milgrom, Astrophys. Jour., 270 (1983), 365, 371, and 384; 333 (1988), 689; 302 (1986), 617; 287 (1984), 571
[78] R. Mosna, I. Hamilton, and L. Delle Site, quant-ph 0504124 and 0511068
[79] S. Nasiri, Symmetry, Integrability, and Geometry: Methods and Application, 2 (2006), 062
[80] S. Nasiri and H. Safari, quant-ph 0505147
[81] S. Nasiri, Y. Sobouti, and F. Taati, quant-ph 0605129
[82] E. Nelson, Quantum fluctuations, Princeton Univ. Press, 1985
[83] H. Nikolić, Eur. Phys. Jour. C, 42 (2005), 365-374 (hep-th 0407228); Found. Phys. Lett., 17 (2004), 363-380 (quant-ph 0208185) and 18 (2005), 123-138 and 549-561 (quant-ph 0406173); hep-th 0601027
[84] H. Nikolić, hep-th 0512186; quant-ph 0512065, 0603207, and 0610138
[85] J. Noldus, gr-qc 0508104
[86] L. Nottale, Fractal spacetime and microphysics, World Scientific, 1993
[87] L. Nottale, Chaos, Solitons, and Fractals, 7 (1996), 877-938; 10 (1999), 459-468; 12 (2000), 1577-1583; 16 (2003), 539-564
[88] H. Ohanian and A. Ruffini, Gravitation and spacetime, Norton, 1994
[89] L. Olavo, Physica A, 262 (1999), 197-214 and 271 (1999), 260-302 and 303-323; Phys. Rev. E, 64 (2001), 036125
[90] R. Parwani, quant-ph 0408185, 0412192, and 0508125
[91] G. Perelman, math.DG 0211159
[92] N. Pinto-Neto, gr-qc 0410001, 0410117, and 0410225
[93] N. Pinto-Neto and E. Santini, gr-qc 0009080 and 0302112; Gener. Relativ. Gravitation, 34 (2002), 505; Phys. Lett. A, 315 (2003), 36; Phys. Rev. D, 59 (1999), 123517 (gr-qc 9811067)
[94] J. Pissondes, Chaos, Solitons, and Fractals, 9 (1998), 1115-1142; hep-th 0009169 and 0010146
[95] S. Popescu, A. Short, and A. Winter, quant-ph 0511225
[96] M. Reginatto, gr-qc 0501030; quant-ph 9909065; cond-mat 9910039; Phys. Rev. A, 58 (1998), 1775-1778
[97] J. Rehacek and Z. Hradil, Jour. Mod. Optics, 15 (2004), 979-982
[98] F. Reif, Fundamentals of statistical and thermal physics, McGraw-Hill, 1965
[99] H. Robertson, Phys. Rev., 46 (1934), 794-801
[100] C. Rovelli, Quantum gravity, Cambridge Univ. Press, 2004
[101] T. Sakaguchi, gr-qc 9704039
[102] E. Santamato, Phys. Rev. D, 29 (1984), 216-222; 32 (1985), 2615-2621; Jour. Math. Phys., 25 (1984), 2477-2480
[103] E. Santamato, Phys. Lett. A, 130 (1988), 199-202
[104] E. Santini, gr-qc 0005092
[105] F. and A. Shojai, gr-qc 0404102 and 0306099; Inter. Jour. Mod. Phys. A, 15 (2000), 1859-1868
[106] F. Shojai and M. Golshani, Inter. Jour. Mod. Phys. A, 13 (1998), 677-693 and 2135-2144
[107] A. and F. Shojai and N. Dadhich, gr-qc 0504137
[108] A. and F. Shojai, Physica Scripta, 64 (2001), 413-416
[109] A. Shojai, Inter. Jour. Mod. Phys. A, 15 (2000), 1757-1771
[110] F. Shojai and A. Shirinifard, gr-qc 0504138
[111] A. and F. Shojai, quant-ph 0109025 and 0609109; gr-qc 0105102, 0306100 and 0311076

[112] F. and A. Shojai and M. Golshani, Mod. Phys. Lett. A, 13 (1998), 2725-2729 and 2915-2922; gr-qc 9903050
[113] F. Shojai, Phys. Rev. D, 60 (1999), 124001
[114] F. and A. Shojai, Class. Quant. Grav., 21 (2004), 1-9; gr-qc 0409020
[115] F. and A. Shojai, gr-qc 0105102, 0109052, and 0409036
[116] A. Shojai, Inter. Jour. Mod. Phys. A, 15 (2000), 1757-1771
[117] Y. Shtanov, quant-ph 9705024; Phys. Rev. D, 54 (1996), 2569-2570
[118] L. Smolin, Phys. Lett. A, 113 (1986), 408-412; quant-ph 0609109
[119] Y. Sobouti and S. Nasiri, Inter. Jour. Mod. Phys. B, 7 (1993), 3255-3273
[120] R. Streater, Lost causes in theoretical physics, Wikipedia
[121] F. Strocchi, Rev. Mod. Phys., 31 (1966), 36-40
[122] M. Tegmark, quant-ph 9709032 and 0302131
[123] F. Tipler, quant-ph 0003146 and 0611245
[124] L. Vaidman, quant-ph 0609006, 0001057, and 0111072
[125] J. Vigier, Phys. Lett. A, 135 (1989), 99
[126] R. Wald, General relativity, Univ. Chicago Press, 1984
[127] D. Wallace, quant-ph 0103092
[128] S. Weinberg, Phys. Rev. Lett., 62 (1989), 485-488
[129] J. Wheeler, Phys. Rev. D, 41 (1990), 431-441
[130] Wikipedia, Many worlds
[131] W. Wood and G. Papini, Found. Phys. Lett., 6 (1993), 207-223
[132] T. Zlosnik, P. Ferreira, and G. Starkman, gr-qc 0606039